Journal of Mathematical Physics, Analysis, Geometry
2020, Vol. 16, No. 2, pp. 91–118
doi: https://doi.org/10.15407/mag16.02.091

On the Correlation Functions of the Characteristic Polynomials of Real Random Matrices with Independent Entries

Ievgenii Afanasiev

The paper is concerned with the correlation functions of the characteristic polynomials of real random matrices with independent entries. The asymptotic behavior of the correlation functions is established in the form of a certain integral over unitary self-dual matrices with respect to the invariant measure. The integral is computed in the case of the second order correlation function. From the obtained asymptotics it is clear that the correlation functions behave like that for the Real Ginibre Ensemble up to a factor depending only on the fourth absolute moment of the common probability law of the matrix entries.

Key words: random matrix theory, Ginibre ensemble, correlation functions of characteristic polynomials, moments of characteristic polynomials, SUSY

Mathematical Subject Classification 2010: 60B20, 15B52

1. Introduction

The ensemble of random matrices with independent entries was introduced by Ginibre in 1965 [19]. To be exact, he introduced a partial case when entries of the matrices have a Gaussian distribution. Anyway, the ensemble appeared to be significant and has been attracting scientists' attention since that time.

Random matrices with independent entries are usually considered over complex numbers, real numbers or quaternions. An asymptotic behavior of the correlation functions of the characteristic polynomials was recently computed in the complex case [2]. The goal of the current paper is to establish a similar result in the real case.

Let us proceed to precise definitions. We consider the matrices of the form

$$M_n = \frac{1}{\sqrt{n}} X = \frac{1}{\sqrt{n}} (x_{jk})_{j,k=1}^n, \tag{1.1}$$

where x_{jk} are i.i.d. real random variables such that

$$\mathbf{E}\{x_{jk}\} = 0 \quad \text{and} \quad \mathbf{E}\{x_{jk}^2\} = 1. \tag{1.2}$$

Here and everywhere below $\mathbf{E}$ denotes the expectation with respect to all random variables. This ensemble has various applications in physics, neuroscience, economics, etc. For detailed information, see [3] and references therein.

Define the Normalized Counting Measure (NCM) of eigenvalues as

$$N_n(\Delta) = \#\{\lambda_j^{(n)} \in \Delta,\ j = 1, \dots, n\}/n,$$

where Δ is an arbitrary Borel set in the complex plane, $\left\{\lambda_j^{(n)}\right\}_{j=1}^n$ are the eigenvalues of M_n. The NCM is known to converge to the uniform distribution on the unit disc. The distribution is called the circular law. This result has a long and rich history. Mehta was the first who obtained it for x_{jk} being complex Gaussian in 1967 [28]. The proof strongly relied on the explicit formula for the common probability density of eigenvalues due to Ginibre [19]. Unfortunately, there is no such a formula in the general case. That is why other methods have to be used. The Hermitization approach introduced by Girko [20] appeared to be an effective method. The main idea is to reduce the study of matrices (1.1) to the study of Hermitian matrices using the logarithmic potential of a measure

$$P_\mu(z) = \int\limits_{\mathbb{C}} \log |z - \zeta|\ d\mu(\zeta).$$

This approach was successfully developed by Girko in the next series of works [21–24]. The final result in the most general case was established by Tao and Vu [39]. Notice that there are a lot of partial results besides those listed above. The interested reader is directed to [5].

The Central Limit Theorem (CLT) for linear statistics of real non-Hermitian random matrices was proven in some partial cases in [26, 30, 31, 40]. The best result for today was obtained by Cipolloni, Erdős and Schröder in [12]. They proved CLT for a bit more than twice differentiable test functions assuming that the common distribution of matrix entries has finite moments. A local regime for matrices (1.1) was studied in [6, 10, 40]. In [6], the k-point correlation function and its asymptotic behavior were computed for the Real Ginibre Ensemble (i.e., if matrix entries are Gaussian, this ensemble is often referred as GinOE similarly to the Gaussian Orthogonal Ensemble (GOE) in the real symmetric case). In [40], it was established that the k-point correlation function converges in vague topology to that for GinOE if x_{jk} has the first four moments as in the Gaussian case. The condition of matching moments was recently overcome at the edge of the spectrum (i.e., $|z| = 1$) in [10].

One can observe that non-Hermitian random matrices are more complicated than their Hermitian counterparts. Indeed, the Hermitian case was successfully dealt with using the Stieltjes transform or the moments method. However, a measure in the plane can not be recovered from its Stieltjes transform or its moments. Thus these approaches to the analysis fail in the non-Hermitian case.

The present paper suggests using the supersymmetry technique (SUSY). It is a rather powerful method which is widely applied at the physical level of rigor (for instance [17, 29]). There are also a lot of rigorous results, which were obtained using SUSY in the recent years, e.g., [11, 13, 14, 33–35] etc. The supersymmetry technique is usually used in order to obtain an integral representation for ratios of determinants. Since the main spectral characteristics such as density of states, spectral correlation functions, etc. often can be expressed via ratios of determinants, SUSY allows one to get the integral representation for these characteristics too. For a detailed discussion on connection between the spectral characteristics and the ratios of determinants, see [7, 25, 38]. See also [18, 32].

Let us consider the second spectral correlation function R_2 defined by the equality

$$\mathbf{E}\left\{2\sum_{1\le j_1<j_2\le n}\eta\left(\lambda_{j_1}^{(n)},\lambda_{j_2}^{(n)}\right)\right\}=\int\limits_{\mathbb{C}^2}\eta(\lambda_1,\lambda_2)R_2(\lambda_1,\lambda_2)d\bar{\lambda}_1d\lambda_1d\bar{\lambda}_2d\lambda_2,$$

where the function $\eta\colon \mathbb{C}^2\to\mathbb{C}$ is bounded, continuous and symmetric in its arguments. Using the logarithmic potential, R_2 can be represented via ratios of the determinants of M_n with the most singular term of the form

$$\int\limits_0^{\varepsilon_0}\int\limits_0^{\varepsilon_0}\frac{\partial^2}{\partial\delta_1\partial\delta_2}\mathbf{E}\left\{\prod_{j=1}^{2}\frac{\det\left((M_n-z_j)(M_n-z_j)^*+\delta_j\right)}{\det\left((M_n-z_j)(M_n-z_j)^*+\varepsilon_j\right)}\right\}\Bigg|_{\delta=\varepsilon}d\varepsilon_1d\varepsilon_2. \tag{1.3}$$

The integral representation for (1.3) obtained by SUSY will contain both commuting and anti-commuting variables. Such type integrals are rather difficult to analyze. Based on this reason, one should study a similar but simpler integral to shed light on the situation. This integral arises from the study of the correlation functions of the characteristic polynomials. Moreover, the correlation functions of the characteristic polynomials are of independent interest. They were studied for many ensembles of Hermitian and real symmetric matrices, for instance, [1, 8, 9, 34, 36, 37] etc.

Let us introduce the m^{th} correlation function of the characteristic polynomials

$$\mathsf{f}_m(Z)=\mathbf{E}\left\{\prod_{j=1}^{m}\det\left(M_n-z_j\right)\left(M_n-z_j\right)^*\right\}, \tag{1.4}$$

where

$$Z=\operatorname{diag}\{z_1,\dots,z_m\} \tag{1.5}$$

and $z_1,\dots,z_m$ are complex parameters which may depend on n. We are interested in the asymptotic behavior of (1.4), as $n\to\infty$, for

$$z_j=z_0+\frac{\zeta_j}{\sqrt{n}},\quad j=1,2,\dots,m, \tag{1.6}$$

where z_0 is either in the bulk ($|z_0|<1$) or at the edge ($|z_0|=1$) of the spectrum and $\zeta_1,\dots,\zeta_m$ are n-independent complex numbers. In the present paper, we confine ourself to the case of real z_0 in the bulk.

In order to formulate the main result of the paper let us introduce some notations. Put

$$J_k = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}, \tag{1.7}$$

where I_k is a unit $k \times k$ matrix. We omit the dimension index when it is clear from the context. For any even size matrix A its dual matrix A^R is defined as follows:

$$A^R = -JA^T J, \tag{1.8}$$

where A^T stands for the transposed matrix. The main result of the paper is

Theorem 1.1. *Let an ensemble of real random matrices M_n be defined by (1.1) and (1.2). Let also the first $2m$ moments of the common distribution of entries of M_n be finite and z_j, $j = 1, \dots, m$, have the form (1.6). Then*

(i) *the m^{th} correlation function of the characteristic polynomials (1.4) satisfies the asymptotic relation*

$$\lim_{n\to\infty} n^{-m^2+m} \frac{\mathsf{f}_m(Z)}{\mathsf{f}_1(z_1)\cdots \mathsf{f}_1(z_m)} = C_{m,z_0} e^{\frac{m^2-m}{2}(1-z_0^2)^2\kappa_4} \times \int\limits_{\substack{V=V^R\\ V\in U(2m)}} \exp\left\{\frac{1}{2}\operatorname{tr}\check{\mathcal{Z}} V \check{\mathcal{Z}}^R V^* - \frac{1}{2}\operatorname{tr}\check{\mathcal{Z}}\check{\mathcal{Z}}^R\right\} d\mu_s(V), \tag{1.9}$$

where C_{m,z_0} is some constant, which does not depend on the common distribution of entries and on $\zeta_1, \dots, \zeta_m$; $\kappa_4 = \mathbf{E}\{x_{11}^4\} - 3$, $U(2m)$ is a unitary group, the probabilistic measure $d\mu_s(V)$ corresponds to the differential form

$$\det{}^{-m+1/2} V \bigwedge_{j,k\le m} dv_{jk} \bigwedge_{j<k\le m} dv_{j,k+m} \wedge dv_{k+m,j} \tag{1.10}$$

and

$$\check{\mathcal{Z}} = \operatorname{diag}\{\bar\zeta_1, \dots, \bar\zeta_m, \zeta_1, \dots, \zeta_m\}. \tag{1.11}$$

(ii) *in the particular case $m = 2$, the integral over self-dual unitary matrices can be computed, and we have*

$$\lim_{n\to\infty} n^{-2} \frac{\mathsf{f}_2(Z)}{\mathsf{f}_1(z_1)\mathsf{f}_1(z_2)} = C_{2,z_0} e^{(1-|z_0|^2)^2\kappa_4} \frac{\operatorname{Pf}(K(\zeta_j,\zeta_k))_{j,k=1}^2}{\triangle(\zeta_1,\zeta_2,\bar\zeta_1,\bar\zeta_2)},$$

where $\triangle(\zeta_1,\zeta_2,\bar\zeta_1,\bar\zeta_2)$ is a Vandermonde determinant of ζ_1, ζ_2, $\bar\zeta_1$, $\bar\zeta_2$, and

$$K(\zeta_j,\zeta_k) = e^{-\frac{|\zeta_j|^2}{2}-\frac{|\zeta_k|^2}{2}} \begin{pmatrix} (\zeta_j-\zeta_k)e^{\zeta_j\zeta_k} & (\zeta_j-\bar\zeta_k)e^{\zeta_j\bar\zeta_k} \\ (\bar\zeta_j-\zeta_k)e^{\bar\zeta_j\zeta_k} & (\bar\zeta_j-\bar\zeta_k)e^{\bar\zeta_j\bar\zeta_k} \end{pmatrix}.$$

Theorem 1.1 shows that the asymptotics of f_2 (here and below we omit Z only if $Z = \operatorname{diag}\{z_1, \dots, z_m\}$) is similar to the asymptotics of the 2-point spectral correlation function (see [6]). Besides, it is naturally to put a conjecture about the form of the asymptotic behavior of f_m for any m.

Set of matrices	Matrix	Column	Entry
$\boldsymbol{Q}$	$Q_{p,s}$		$q_{\alpha\beta}^{(p,s)}$
$\boldsymbol{\Xi}$	$\Xi_{p,s}$		$\xi_{\alpha\beta}^{(p,s)}$
	Φ	$\boldsymbol{\phi}_j$	ϕ_{kj}
	Θ	$\boldsymbol{\theta}_j$	θ_{kj}
	$Y_{k,p,s}$		$y_{\alpha\beta}^{(k,p,s)}$
	U		u_{kj}
	V		v_{kj}

Table 1.1: Notation correspondence

Conjecture 1.2. *In the setting of Theorem 1.1 we expect that for any m*

$$\lim_{n\to\infty} n^{-m^2+m}\frac{\mathsf{f}_m(Z)}{\mathsf{f}_1(z_1)\cdots\mathsf{f}_1(z_m)} = C_{m,z_0}e^{\frac{m^2-m}{2}(1-z_0^2)^2\kappa_4}\frac{\mathrm{Pf}(K(\zeta_j,\zeta_k))_{j,k=1}^m}{\triangle(\zeta_1,\dots,\zeta_m,\bar{\zeta}_1,\dots,\bar{\zeta}_m)}.$$

The paper is organized as follows. Section 2 is devoted to the derivation of the suitable integral representation for f_m by using the SUSY approach. In Section 3, we apply the steepest descent method to the obtained integral representation and find out the asymptotic behavior of f_m. For the reader's convenience, the latter section is divided into two parts treating the Gaussian and the general cases respectively.

1.1. Notations. Throughout the paper, lower-case letters denote the scalars, bold lower-case letters denote the vectors, upper-case letters denote the matrices and bold upper-case letters denote the sets of matrices. We use the same letter for a matrix, for its columns and for its entries. Table 1.1 shows an exact correspondence. Besides, for any matrix A we denote by $(A)_j$ its j-th column and by $(A)_{kj}$, its entry in the k-th row and in the j-th column.

The term "Grassmann variable" is a synonym for "anti-commuting variable". The variables of integration ϕ, φ, θ, ϑ, ρ, ξ, τ and ν are Grassmann variables, all the other variables of integration unspecified by an integration domain are either complex or real. We split all the generators of the Grassmann algebra into two equal sets and consider the generators from the second set as "conjugates" of those from the first set. I.e., for the Grassmann variable υ we use υ^* to denote its "conjugate". Furthermore, if $\Upsilon = (\upsilon_{jk})$ means a matrix of Grassmann variables, then Υ^+ is a matrix (υ_{kj}^*). The d-dimensional vectors are identified with $d \times 1$ matrices.

The integrals without limits denote either an integration over Grassmann variables or an integration over the whole space $\mathbb{C}^d$ or $\mathbb{R}^d$. Let also $d\boldsymbol{t}^*d\boldsymbol{t}$ ($\boldsymbol{t} = (t_1,\dots,t_d)^T \in \mathbb{C}^d$) denote the measure $\prod_{j=1}^d d\bar{t}_j dt_j$ on the space $\mathbb{C}^d$. Similarly, for the vectors with anti-commuting entries $d\boldsymbol{\tau}^+ d\boldsymbol{\tau} = \prod_{j=1}^d d\tau_j^* d\tau_j$. Note that the space of matrices is a linear space over $\mathbb{C}$. Thus the same notations are used for matrices as well.

$\langle\cdot,\cdot\rangle$ denotes a standard scalar product on $\mathbb{C}^d$. For matrices, $\langle A, B\rangle = \operatorname{tr} B^*A$. For sets of matrices, $\langle \boldsymbol{A}, \boldsymbol{B}\rangle = \sum_j \langle A_j, B_j\rangle$.

$\binom{m}{p} \times \binom{m}{s}$ matrices appear in the statement of Proposition 2.1. It is natural to number rows and columns of such matrices by subsets of an m-element set. To this end, set

$$\mathcal{I}_{m,p'} = \{\alpha \in \mathbb{Z}^{p'} \mid 1 \le \alpha_1 < \ldots < \alpha_{p'} \le m\}. \tag{1.12}$$

If $p' = 0$, we define $\mathcal{I}_{m,p'}$ as $\{\varnothing\}$.

Throughout the paper, $U(m)$, $O(m)$, $\mathrm{USp}(m)$ denote the groups of unitary $m\times m$ matrices, orthogonal $m\times m$ matrices, unitary symplectic $2m\times 2m$ matrices. μ denotes a corresponding Haar measure. In addition, C, C_1 denote various n-independent constants which can be different in different formulas.

2. Integral representation for f_m

In this section, we obtain a convenient integral representation for the correlation function of the characteristic polynomials f_m defined by (1.4).

Proposition 2.1. *Let an ensemble M_n be defined by (1.1) and (1.2). Then the m^{th} correlation function of the characteristic polynomials f_m defined by (1.4) can be represented in the following form:*

$$\mathsf{f}_m = \left(\frac{n}{\pi}\right)^{c_m} \int g(\boldsymbol{Q}) e^{(n-c_m)f(\boldsymbol{Q})} d\boldsymbol{Q}, \tag{2.1}$$

where $c_m = 2^{2m-1}$, $\boldsymbol{Q} = (\boldsymbol{Q}_j)_{j=0}^m$, $\boldsymbol{Q}_j = \{Q_{p,s} \mid p+s = 2j,\ 0 \le p, s \le m\}$, $Q_{p,s}$ is a complex $\binom{m}{p} \times \binom{m}{s}$ matrix, $d\boldsymbol{Q} = \prod\limits_{\substack{p+s \text{ is even} \\ 0\le p,s\le m}} dQ^*_{p,s} dQ_{p,s}$ and

$$\begin{aligned} f(\boldsymbol{Q}) &= -\langle \boldsymbol{Q}, \boldsymbol{Q}\rangle + \log h(\boldsymbol{Q}); && (2.2)\\ g(\boldsymbol{Q}) &= \left(h(\boldsymbol{Q})^{c_m} + n^{-1/2}\mathtt{p}_a(\boldsymbol{Q})\right) \exp\left\{-c_m\langle \boldsymbol{Q}, \boldsymbol{Q}\rangle\right\};\\ h(\boldsymbol{Q}) &= \operatorname{Pf} F + n^{-1/2}\widetilde{h}(\boldsymbol{Q}_2) + n^{-1}\mathtt{p}_c(\widehat{\boldsymbol{Q}}); && (2.3)\end{aligned}$$

$$F = \begin{pmatrix} B_{2,0} & 0 & -Z & Q_1 \\ 0 & B^*_{0,2} & -Q^*_1 & -Z^* \\ Z & \overline{Q}_1 & B^*_{2,0} & 0 \\ -Q^T_1 & Z^* & 0 & B_{0,2} \end{pmatrix}; \tag{2.4}$$

$$(B_{2,0})_{\alpha_1\alpha_2} = -q^{(2,0)}_{\alpha\varnothing}, \quad (B_{0,2})_{\alpha_1\alpha_2} = -q^{(0,2)}_{\varnothing\alpha}, \quad \alpha \in \mathcal{I}_{m,2},$$

with $\mathtt{p}_a(\boldsymbol{Q})$, $\mathtt{p}_c(\widehat{\boldsymbol{Q}})$ and $\widetilde{h}(\boldsymbol{Q}_2)$ being certain polynomials specified in the proof below, $\widehat{\boldsymbol{Q}}$ containing all $\boldsymbol{Q}_j$ except $\boldsymbol{Q}_1$, and $\mathcal{I}_{m,2}$ defined in (1.12).

Remark 2.2. Let us consider the transformations

$$F = \begin{pmatrix} B_{2,0} & 0 & -Z & Q_1 \\ 0 & B_{0,2}^* & -Q_1^* & -Z^* \\ Z & \overline{Q}_1 & B_{2,0}^* & 0 \\ -Q_1^T & Z^* & 0 & B_{0,2} \end{pmatrix} \sim \begin{pmatrix} Z^* & 0 & B_{0,2} & -Q_1^T \\ 0 & -Z & Q_1 & B_{2,0} \\ B_{0,2}^* & -Q_1^* & -Z^* & 0 \\ \overline{Q}_1 & B_{2,0}^* & 0 & Z \end{pmatrix}$$

$$\sim \begin{pmatrix} Z^* & 0 & B_{0,2} & -Q_1^T \\ 0 & Z & Q_1 & B_{2,0} \\ -B_{0,2}^* & -Q_1^* & Z^* & 0 \\ \overline{Q}_1 & -B_{2,0}^* & 0 & Z \end{pmatrix} =: \begin{pmatrix} \check{Z} & \check{Q} \\ -\check{Q}^* & \check{Z} \end{pmatrix} =: \check{F}, \tag{2.5}$$

where $\check{Q}$ and $\check{Z}$ are the $2m \times 2m$ matrices. Notice that $\det \check{F} = \det F$, because the first transformation in (2.5) is a permutation of lines and columns, and the second one is a sign change. Moreover, $\frac{1}{2} \operatorname{tr} \check{Q}^* \check{Q} = \operatorname{tr} Q_{0,2}^* Q_{0,2} + \operatorname{tr} Q_{1,1}^* Q_{1,1} + \operatorname{tr} Q_{2,0}^* Q_{2,0}$. Thus one can replace $\boldsymbol{Q}_1$ by $\check{Q}$ and $\operatorname{Pf} F$ by $\det^{1/2} \check{F}$ in the assertion of Proposition 2.1.

Remark 2.3. There is a well-known fact from the matrix theory that any skew-symmetric matrix can be block-diagonalized with a unitary matrix. In our case, this fact implies that $\check{Q} = U \check{\Lambda} U^T$, where

$$\check{\Lambda} = \operatorname{diag} \{\lambda_j L\}_{j=1}^m, \ \lambda_j \ge 0, \ L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad U \in U(2m).$$

Permuting lines and columns of $\check{\Lambda}$ and changing U in a proper way, one can assume that $\check{\Lambda}$ has the form

$$\check{\Lambda} = \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix}, \ \Lambda = \operatorname{diag} \{\lambda_j\}_{j=1}^m.$$

In order to perform an asymptotic analysis let us change the variables $\check{Q} = U \check{\Lambda} U^T$ in (2.1). Then the Jacobian is $\frac{2^m \pi^{m^2}}{\left(\prod_{j=1}^{m-1} j!\right)^2} \triangle^4(\Lambda^2) \prod\limits_{j=1}^m \lambda_j$. We obtain

$$\begin{aligned} \mathsf{f}_m = C n^{c_m} \int\limits_{\mathcal{D}} \triangle^4(\Lambda^2) \prod_{j=1}^m \lambda_j \times \left[g_0(\check{\Lambda}, \widehat{\boldsymbol{Q}}) + \frac{1}{\sqrt{n}} g_r(U \check{\Lambda} U^T, \widehat{\boldsymbol{Q}}) \right] \\ \times \exp \left\{ (n - c_m) \left[f_0(\check{\Lambda}, \widehat{\boldsymbol{Q}}) + \frac{1}{\sqrt{n}} f_r(U \check{\Lambda} U^T, \widehat{\boldsymbol{Q}}) \right] \right\} d\mu(U) d\Lambda d\widehat{\boldsymbol{Q}}, \end{aligned} \tag{2.6}$$

where $\mathcal{D} = \{(\Lambda, U, \widehat{\boldsymbol{Q}}) \mid \lambda_j \ge 0, \, j = 1, \ldots, m, \, U \in U(2m)\}$, μ is a Haar measure, $d\Lambda = \prod\limits_{j=1}^m d\lambda_j$ and

$$\begin{aligned} f_0(\boldsymbol{Q}) &= -\langle \boldsymbol{Q}, \boldsymbol{Q} \rangle + \log h_0(\check{Q}); \\ g_0(\boldsymbol{Q}) &= h_0(\check{Q})^{c_m} \exp\{-c_m \langle \boldsymbol{Q}, \boldsymbol{Q} \rangle\} = e^{c_m f_0(\boldsymbol{Q})}; \end{aligned} \tag{2.7}$$

$$h_0(\check{Q}) = \det{}^{1/2}\begin{pmatrix} z_0 I_{2m} & \check{Q} \\ -\check{Q}^* & z_0 I_{2m} \end{pmatrix} = \prod_{j=1}^{m}(z_0^2 + \lambda_j^2); \tag{2.8}$$

$$f_r(\boldsymbol{Q}) = \sqrt{n}(f(\boldsymbol{Q}) - f_0(\boldsymbol{Q})); \tag{2.9}$$
$$g_r(\boldsymbol{Q}) = \sqrt{n}(g(\boldsymbol{Q}) - g_0(\boldsymbol{Q})).$$

Notice that $f_0(U\check{\Lambda}U^T, \widehat{\boldsymbol{Q}}) = f_0(\check{\Lambda}, \widehat{\boldsymbol{Q}})$ and the same is for g_0.

Remark 2.4. In the special case $m = 1$, we have

$$\mathsf{f}_1(z) = \frac{n}{\pi}\int \exp\left\{n(-|q|^2 + \log(|z|^2 + |q|^2))\right\} d\bar{q}dq.$$

Changing the variables to polar coordinates and performing a simple Laplace integration, we obtain

$$\mathsf{f}_1(z) = 2n\int_0^{+\infty} r\exp\left\{n(-r^2 + \log(|z|^2 + r^2))\right\} dr$$
$$= \sqrt{2\pi n}\, e^{n(|z|^2-1)}(1 + o(1)). \tag{2.10}$$

Remark 2.5. In the real Gaussian case, representations (2.1) and (2.6) become much more simple and have the form

$$\mathsf{f}_m = \left(\frac{n}{\pi}\right)^{2m^2-m}\int e^{nf(\check{Q})} d\check{Q}^* d\check{Q}$$
$$= Cn^{2m^2-m}\int_{\mathbb{R}_+^m}\int_{U(m)} \triangle^4(\Lambda^2)\prod_{j=1}^{m}\lambda_j \times e^{nf(U\check{\Lambda}U^T)} d\mu(U)d\Lambda, \tag{2.11}$$

where

$$f(\check{Q}) = -\frac{1}{2}\operatorname{tr}\check{Q}^*\check{Q} + \frac{1}{2}\log\det\check{F} \tag{2.12}$$

and $\check{Q}$, $\check{F}$ are defined in (2.5).

2.1. Proof of Proposition 2.1. The proof is strongly relied on the SUSY techniques. A reader who is not familiar with Grassmann variables can find all the necessary facts in [15] or [16]. For more serious introduction to SUSY, see [4].

The key formulas of the subsection are the well-known Gaussian integration formulas

$$\int_{\mathbb{C}^n} \exp\{-\boldsymbol{t}^* A\boldsymbol{t} - \boldsymbol{t}^*\boldsymbol{h}_2 - \boldsymbol{h}_1^*\boldsymbol{t}\}\, d\boldsymbol{t}^* d\boldsymbol{t} = \pi^n \det{}^{-1} A\exp\{\boldsymbol{h}_1^* A^{-1}\boldsymbol{h}_2\}, \tag{2.13}$$

$$\int_{\mathbb{R}^n} \exp\left\{-\frac{1}{2}\boldsymbol{t}^T A\boldsymbol{t}\right\} d\boldsymbol{t} = (2\pi)^{n/2}\det{}^{-1/2} A, \tag{2.14}$$

valid for any positive definite matrix A and even Grassmann variable vectors (i.e., vectors whose components are sums of products of even number of Grassmann variables) $\boldsymbol{h}_1$, $\boldsymbol{h}_2$, and its Grassmann analogs

$$\int \exp\left\{-\boldsymbol{\tau}^+ A\boldsymbol{\tau} - \boldsymbol{\tau}^+\boldsymbol{v}_2 - \boldsymbol{v}_1^+\boldsymbol{\tau}\right\} d\boldsymbol{\tau}^+ d\boldsymbol{\tau} = \det A \exp\{\boldsymbol{v}_1^+ A^{-1}\boldsymbol{v}_2\}, \tag{2.15}$$

$$\int \exp\left\{-\frac{1}{2}\boldsymbol{\tau}^T A\boldsymbol{\tau}\right\} d\boldsymbol{\tau} = \operatorname{Pf} A. \tag{2.16}$$

(2.15) is valid for an arbitrary complex matrix A and odd Grassmann variable vectors (i.e., vectors whose components are sums of products of odd number of Grassmann variables) $\boldsymbol{v}_1^+$, $\boldsymbol{v}_2$, whereas (2.16) is valid for any complex skew-symmetric matrix A. Rewrite the expression (1.4) for f_m using (2.15) and (1.1),

$$\mathsf{f}_m = \mathbf{E}\Bigg\{\int \exp\Bigg\{-\sum_{j=1}^m \boldsymbol{\phi}_j^+\left(\frac{1}{\sqrt{n}}X - z_j\right)\boldsymbol{\phi}_j - \sum_{j=1}^m \boldsymbol{\theta}_j^+\left(\frac{1}{\sqrt{n}}X - z_j\right)^*\boldsymbol{\theta}_j\Bigg\} d\Phi d\Theta\Bigg\},$$

where $\boldsymbol{\phi}_j$, $\boldsymbol{\theta}_j$, $j = 1, \ldots, m$ are n-dimensional vectors whose components are ϕ_{kj} and θ_{kj} respectively, $d\Phi = \prod\limits_{j=1}^m d\boldsymbol{\phi}_j^+ d\boldsymbol{\phi}_j$ and $d\Theta = \prod\limits_{j=1}^m d\boldsymbol{\theta}_j^+ d\boldsymbol{\theta}_j$. The terms in the exponent can be rearranged as follows:

$$-\sum_{j=1}^m \boldsymbol{\phi}_j^+ X\boldsymbol{\phi}_j = -\operatorname{tr}\Phi^+ X\Phi = \operatorname{tr}\Phi\Phi^+ X = \sum_{k,l=1}^n (\Phi\Phi^+)_{lk} x_{kl},$$

$$-\sum_{j=1}^m \boldsymbol{\theta}_j^+ X^*\boldsymbol{\theta}_j = -\operatorname{tr}\Theta^+ X^*\Theta = \operatorname{tr}\Theta\Theta^+ X^* = \sum_{k,l=1}^n (\Theta\Theta^+)_{kl}\bar{x}_{kl},$$

$$\sum_{j=1}^m \boldsymbol{\phi}_j^+ z_j\boldsymbol{\phi}_j = \sum_{j=1}^m\sum_{k=1}^n \phi_{kj}^* z_j\phi_{kj} = \sum_{k=1}^n\sum_{j=1}^m \phi_{kj}^* z_j\phi_{kj} = \sum_{k=1}^n \boldsymbol{\varphi}_k^+ Z\boldsymbol{\varphi}_k,$$

$$\sum_{j=1}^m \boldsymbol{\theta}_j^+ \bar{z}_j\boldsymbol{\theta}_j = \sum_{j=1}^m\sum_{k=1}^n \theta_{kj}^* \bar{z}_j\theta_{kj} = \sum_{k=1}^n\sum_{j=1}^m \theta_{kj}^* \bar{z}_j\theta_{kj} = \sum_{k=1}^n \boldsymbol{\vartheta}_k^+ Z^*\boldsymbol{\vartheta}_k,$$

where Θ and Φ are matrices composed of columns $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m$ and $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_m$, respectively, $\boldsymbol{\varphi}_k = (\Phi^T)_k$, $\boldsymbol{\vartheta}_k = (\Theta^T)_k$, Z is defined in (1.5). Hence,

$$\mathsf{f}_m = \mathbf{E}\Bigg\{\int \exp\Bigg\{\sum_{k=1}^n \boldsymbol{\varphi}_k^+ Z\boldsymbol{\varphi}_k + \sum_{k=1}^n \boldsymbol{\vartheta}_k^+ Z^*\boldsymbol{\vartheta}_k + \frac{1}{\sqrt{n}}\sum_{k,l=1}^n (\Phi\Phi^+)_{lk} x_{kl} + \frac{1}{\sqrt{n}}\sum_{k,l=1}^n (\Theta\Theta^+)_{kl}\bar{x}_{kl}\Bigg\} d\Phi d\Theta\Bigg\}. \tag{2.17}$$

Let us introduce a notation for a kind of the "Laplace–Fourier transform"

$$\psi(t_1, t_2) := \mathbf{E}\left\{e^{t_1 x_{11} + t_2\bar{x}_{11}}\right\}.$$

Then the expectation in (2.17) can be written in the form

$$\begin{aligned}\mathsf{f}_m &= \int \prod_{k,l=1}^{n} \psi\left(\frac{1}{\sqrt{n}}(\Phi\Phi^+)_{lk}, \frac{1}{\sqrt{n}}(\Theta\Theta^+)_{kl}\right) \\ &\quad \times \exp\left\{\sum_{k=1}^{n} \boldsymbol{\varphi}_k^+ Z \boldsymbol{\varphi}_k + \sum_{k=1}^{n} \boldsymbol{\vartheta}_k^+ Z^* \boldsymbol{\vartheta}_k\right\} d\Phi d\Theta \\ &= \int \exp\left\{\sum_{k=1}^{n} \boldsymbol{\varphi}_k^+ Z \boldsymbol{\varphi}_k + \sum_{k=1}^{n} \boldsymbol{\vartheta}_k^+ Z^* \boldsymbol{\vartheta}_k\right. \\ &\quad \left. + \sum_{k,l=1}^{n} \log\psi\left(\frac{1}{\sqrt{n}}(\Phi\Phi^+)_{lk}, \frac{1}{\sqrt{n}}(\Theta\Theta^+)_{kl}\right)\right\} d\Phi d\Theta.\end{aligned}$$

Expansion of $\log\psi$ into series gives us

$$\begin{aligned}\mathsf{f}_m &= \int \exp\left\{\sum_{k=1}^{n} \boldsymbol{\varphi}_k^+ Z \boldsymbol{\varphi}_k + \sum_{k=1}^{n} \boldsymbol{\vartheta}_k^+ Z^* \boldsymbol{\vartheta}_k\right. \\ &\quad \left. + \sum_{k,l=1}^{n} \sum_{p,s=0}^{m} \frac{\kappa_{p,s}}{p!s!} \frac{1}{n^{(p+s)/2}} \left((\Phi\Phi^+)_{lk}\right)^p \left((\Theta\Theta^+)_{kl}\right)^s\right\} d\Phi d\Theta, \end{aligned} \tag{2.18}$$

with

$$\kappa_{p,s} = \frac{\partial^{p+s}}{\partial^p t_1 \partial^s t_2} \log\psi(t_1, t_2)\Big|_{t_1=t_2=0}. \tag{2.19}$$

In particular,

$$\begin{aligned}&\kappa_{0,0} = 0; \\ &\kappa_{1,0} = \overline{\kappa_{0,1}} = \mathbf{E}\{x_{11}\} = 0; \\ &\kappa_{2,0} = \overline{\kappa_{0,2}} = \mathbf{E}\{x_{11}^2\} - \overset{2}{\mathbf{E}}\{x_{11}\} = \mathbf{E}\{x_{11}^2\}; \\ &\kappa_{1,1} = \mathbf{E}\{|x_{11}|^2\} - |\mathbf{E}\{x_{11}\}|^2 = 1.\end{aligned} \tag{2.20}$$

Let us transform the terms in the exponent again

$$\begin{aligned}&\sum_{k,l=1}^{n} \left((\Phi\Phi^+)_{lk}\right)^p \left((\Theta\Theta^+)_{kl}\right)^s \\ &\quad = \sum_{k,l=1}^{n} \left(\sum_{j=1}^{m} \phi_{lj}\phi_{kj}^*\right)^p \left(\sum_{j=1}^{m} \theta_{kj}\theta_{lj}^*\right)^s \\ &\quad = p!s! \sum_{k,l=1}^{n} \sum_{\substack{\alpha\in\mathcal{I}_{m,p} \\ \beta\in\mathcal{I}_{m,s}}} \prod_{q=1}^{p} \phi_{l\alpha_q}\phi_{k\alpha_q}^* \prod_{r=1}^{s} \theta_{k\beta_r}\theta_{l\beta_r}^* \\ &\quad = (-1)^{p^2} p!s! \sum_{k,l=1}^{n} \sum_{\substack{\alpha\in\mathcal{I}_{m,p} \\ \beta\in\mathcal{I}_{m,s}}} \prod_{r=s}^{1} \theta_{k\beta_r} \prod_{q=p}^{1} \phi_{k\alpha_q}^* \prod_{q=1}^{p} \phi_{l\alpha_q} \prod_{r=1}^{s} \theta_{l\beta_r}^*\end{aligned}$$

$$= p!s! \sum_{\substack{\alpha\in\mathcal{I}_{m,p}\\ \beta\in\mathcal{I}_{m,s}}} \left(\sum_{k=1}^{n}(-1)^p \prod_{r=s}^{1}\theta_{k\beta_r}\prod_{q=p}^{1}\phi^*_{k\alpha_q}\right)\left(\sum_{k=1}^{n}\prod_{q=1}^{p}\phi_{k\alpha_q}\prod_{r=1}^{s}\theta^*_{k\beta_r}\right), \quad (2.21)$$

where $\mathcal{I}_{m,p'}$ is defined in (1.12).

At this point the Hubbard–Stratonovich transformation is applied. The transformation is an employment of (2.13) or (2.15) in the reverse direction. It yields for even $p+s$,

$$\exp\left\{\kappa_{p,s}n^{-(p+s)/2}\left(\sum_{k=1}^{n}(-1)^p \prod_{r=s}^{1}\theta_{k\beta_r}\prod_{q=p}^{1}\phi^*_{k\alpha_q}\right)\left(\sum_{k=1}^{n}\prod_{q=1}^{p}\phi_{k\alpha_q}\prod_{r=1}^{s}\theta^*_{k\beta_r}\right)\right\}$$
$$= \frac{n}{\pi}\int \exp\left\{-n^{-\frac{p+s-2}{4}}\sum_{k=1}^{n}\widetilde{y}^{(k,p,s)}_{\beta\alpha}q^{(p,s)}_{\alpha\beta}\right.$$
$$\left. - n^{-\frac{p+s-2}{4}}\sum_{k=1}^{n}\bar{q}^{(p,s)}_{\alpha\beta}y^{(k,p,s)}_{\alpha\beta} - n\left|q^{(p,s)}_{\alpha\beta}\right|^2\right\} d\bar{q}^{(p,s)}_{\alpha\beta}dq^{(p,s)}_{\alpha\beta}, \quad (2.22)$$

where

$$\widetilde{y}^{(k,p,s)}_{\beta\alpha} = \sqrt{\kappa_{p,s}}(-1)^p\prod_{r=s}^{1}\theta_{k\beta_r}\prod_{q=p}^{1}\phi^*_{k\alpha_q}; \quad (2.23)$$

$$y^{(k,p,s)}_{\alpha\beta} = \sqrt{\kappa_{p,s}}\prod_{q=1}^{p}\phi_{k\alpha_q}\prod_{r=1}^{s}\theta^*_{k\beta_r}. \quad (2.24)$$

Here and below we take a branch of the square root such that its argument is in $[0,\pi)$. Similarly, for odd $p+s$ we have

$$\exp\left\{\kappa_{p,s}n^{-(p+s)/2}\left(\sum_{k=1}^{n}(-1)^p \prod_{r=1}^{s}\theta_{k\beta_r}\prod_{q=1}^{p}\phi^*_{k\alpha_q}\right)\left(\sum_{k=1}^{n}\prod_{q=1}^{p}\phi_{k\alpha_q}\prod_{r=1}^{s}\theta^*_{k\beta_r}\right)\right\}$$
$$= \int \exp\left\{ - n^{-\frac{p+s}{4}}\sum_{k=1}^{n}\widetilde{y}^{(k,p,s)}_{\beta\alpha}\xi^{(p,s)}_{\alpha\beta} - n^{-\frac{p+s}{4}}\sum_{k=1}^{n}\left(\xi^{(p,s)}_{\alpha\beta}\right)^* y^{(k,p,s)}_{\alpha\beta}\right.$$
$$\left. - \left(\xi^{(p,s)}_{\alpha\beta}\right)^*\xi^{(p,s)}_{\alpha\beta}\right\} d\left(\xi^{(p,s)}_{\alpha\beta}\right)^* d\xi^{(p,s)}_{\alpha\beta}. \quad (2.25)$$

Then the combination of (2.18), (2.21), (2.22) and (2.25) gives us

$$\mathsf{f}_m = \left(\frac{n}{\pi}\right)^{c_m}\int\prod_{k=1}^{n}\mathsf{j}_k \prod_{\substack{p+s \text{ is odd}\\ 0\le p,s\le m}} e^{-\operatorname{tr}\Xi^+_{p,s}\Xi_{p,s}}d\Xi^+_{p,s}d\Xi_{p,s}$$
$$\times \prod_{\substack{p+s \text{ is even}\\ 0\le p,s\le m}} e^{-n\operatorname{tr}Q^*_{p,s}Q_{p,s}}dQ^*_{p,s}dQ_{p,s} \quad (2.26)$$

where

$$\begin{aligned} \mathrm{j}_k = \int \exp\Big\{ b_{k,2} + n^{-1/2} b_{k,4} + n^{-3/4} \mathrm{p}_a^{(1)}(\boldsymbol{\Xi}, \Phi, \Theta) \\ + n^{-1} \mathrm{p}_c^{(1)}(\widehat{\boldsymbol{Q}}, \Phi, \Theta) \Big\} \times d\boldsymbol{\varphi}_k^+ d\boldsymbol{\varphi}_k d\boldsymbol{\vartheta}_k^+ d\boldsymbol{\vartheta}_k, \end{aligned} \tag{2.27}$$

$$b_{k,2} = -\sum_{p+s=2} \left(\operatorname{tr} \widetilde{Y}_{k,p,s} Q_{p,s} + \operatorname{tr} Q_{p,s}^* Y_{k,p,s} \right) + \boldsymbol{\varphi}_k^+ Z \boldsymbol{\varphi}_k + \boldsymbol{\vartheta}_k^+ Z^* \boldsymbol{\vartheta}_k,$$

$$b_{k,4} = -\sum_{p+s=4} \left(\operatorname{tr} \widetilde{Y}_{k,p,s} Q_{p,s} + \operatorname{tr} Q_{p,s}^* Y_{k,p,s} \right), \tag{2.28}$$

$$\mathrm{p}_a^{(1)}(\boldsymbol{\Xi}, \Phi, \Theta) = -\sum_{j=2}^{m} n^{-(j-2)/2} \sum_{p+s=2j-1} \left(\operatorname{tr} \widetilde{Y}_{k,p,s} \Xi_{p,s} + \operatorname{tr} \Xi_{p,s}^+ Y_{k,p,s} \right),$$

$$\mathrm{p}_c^{(1)}(\widehat{\boldsymbol{Q}}, \Phi, \Theta) = -\sum_{j=3}^{m} n^{-(j-3)/2} \sum_{p+s=2j} \left(\operatorname{tr} \widetilde{Y}_{k,p,s} Q_{p,s} + \operatorname{tr} Q_{p,s}^* Y_{k,p,s} \right).$$

In the formulas above, $\Xi_{p,s}$, $Q_{p,s}$, $\widetilde{Y}_{k,p,s}$ and $Y_{k,p,s}$ are matrices whose entries are $\xi_{\alpha\beta}^{(p,s)}$, $q_{\alpha\beta}^{(p,s)}$, $\widetilde{y}_{\beta\alpha}^{(k,p,s)}$ and $y_{\alpha\beta}^{(k,p,s)}$, respectively. The rows and columns are indexed by the elements of the set $\mathcal{I}_{m,p}$ for corresponding p (or s) in lexicographical order. Note also that $\mathrm{p}_a^{(1)}$ and $\mathrm{p}_c^{(1)}$ are the first degree homogeneous polynomials of the entries of $\boldsymbol{\Xi}$ and $\widehat{\boldsymbol{Q}}$, respectively, where $\widehat{\boldsymbol{Q}}$ contains all the $\boldsymbol{Q}_j$ except $\boldsymbol{Q}_1$. One more thing we need is that all the monomials of $\mathrm{p}_a^{(1)}$ have odd degree with respect to $\boldsymbol{\varphi}_k$ and $\boldsymbol{\vartheta}_k$, and all the monomials of $\mathrm{p}_c^{(1)}$ have even degree with respect to $\boldsymbol{\varphi}_k$ and $\boldsymbol{\vartheta}_k$.

Fortunately, the integral in (2.26) over Φ and Θ factorizes. Therefore the integration can be performed over $\boldsymbol{\varphi}_k$ and $\boldsymbol{\vartheta}_k$ separately for every k. Lemma 2.6 provides a corresponding result.

Lemma 2.6. *Let* j_k *be defined by* (2.27). *Then*

$$\mathrm{j}_k = \operatorname{Pf} F + n^{-1/2} \widetilde{h}(\boldsymbol{Q}_2) + n^{-1} \mathrm{p}_c(\widehat{\boldsymbol{Q}}) + n^{-3/2} \mathrm{p}_a^{(2)}(\boldsymbol{\Xi}, \boldsymbol{Q}), \tag{2.29}$$

where F *is defined in* (2.4),

$$\widetilde{h}(\boldsymbol{Q}_2) = \int b_{k,4} e^{b_{k,2}} d\boldsymbol{\varphi}_k^+ d\boldsymbol{\varphi}_k d\boldsymbol{\vartheta}_k^+ d\boldsymbol{\vartheta}_k, \tag{2.30}$$

$\mathrm{p}_c(\widehat{\boldsymbol{Q}})$ *and* $\mathrm{p}_a^{(2)}(\boldsymbol{\Xi}, \boldsymbol{Q})$ *are polynomials such that*

(i) $\mathrm{p}_c(0) = 0$,

(ii) *every monomial of* $\mathrm{p}_a^{(2)}$ *has at least second degree with respect to* $\boldsymbol{\Xi}$.

Proof. The integral j_k is computed by the expansion of the exponent into series. We start with

$$\mathrm{j}_k = \int \Bigl(1 + \sum_{1\le k\le 4m/3} n^{-3k/4}(\mathrm{p}_a^{(1)}(\boldsymbol{\Xi}, \Phi, \Theta))^k\Bigr) \times e^{b_{k,2}+n^{-1/2}b_{k,4}+n^{-1}\mathrm{p}_c^{(1)}(\widehat{\boldsymbol{Q}},\Phi,\Theta)} d\boldsymbol{\varphi}_k^+ d\boldsymbol{\varphi}_k d\boldsymbol{\vartheta}_k^+ d\boldsymbol{\vartheta}_k, \quad (2.31)$$

where the terms of degree higher than $4m$ with respect to $\boldsymbol{\varphi}_k$ and $\boldsymbol{\vartheta}_k$ vanish, because the square of any anti-commuting variable is zero. The monomials of odd degree with respect to $\boldsymbol{\varphi}_k$ and $\boldsymbol{\vartheta}_k$ also vanish after integration. Indeed, for every odd degree homogeneous polynomial $\widetilde{\mathrm{p}}$ the expansion of $\widetilde{\mathrm{p}}\,(\boldsymbol{\varphi}_k, \boldsymbol{\vartheta}_k)\, e^{b_{k,2}+n^{-1/2}b_{k,4}+n^{-1}\mathrm{p}_c(\widehat{\boldsymbol{Q}},\Phi,\Theta)}$ into series gives us only odd degree terms. Whereas the number of Grassmann variables is even, there are no top degree monomials and the integral is zero. Thus (2.31) simplifies to

$$\mathrm{j}_k = \int \Bigl(1 + n^{-3/2}\mathrm{p}_a^{(3)}(\boldsymbol{\Xi}, \Phi, \Theta)\Bigr)\, e^{b_{k,2}+n^{-1/2}b_{k,4}+n^{-1}\mathrm{p}_c^{(1)}(\widehat{\boldsymbol{Q}},\Phi,\Theta)} \times d\boldsymbol{\varphi}_k^+ d\boldsymbol{\varphi}_k d\boldsymbol{\vartheta}_k^+ d\boldsymbol{\vartheta}_k, \quad (2.32)$$

where $\mathrm{p}_a^{(3)}(\boldsymbol{\Xi}, \Phi, \Theta)$ is a polynomial and its every monomial has a degree at least 2 with respect to $\boldsymbol{\Xi}$ and at least 2 with respect to $\boldsymbol{\varphi}_k$ and $\boldsymbol{\vartheta}_k$. Put

$$\mathrm{p}_a^{(2)}(\boldsymbol{\Xi}, \boldsymbol{Q}) := \int \mathrm{p}_a^{(3)}(\boldsymbol{\Xi}, \Phi, \Theta) e^{b_{k,2}+n^{-1/2}b_{k,4}+n^{-1}\mathrm{p}_c^{(1)}(\widehat{\boldsymbol{Q}},\Phi,\Theta)} \times d\boldsymbol{\varphi}_k^+ d\boldsymbol{\varphi}_k d\boldsymbol{\vartheta}_k^+ d\boldsymbol{\vartheta}_k. \quad (2.33)$$

Note that $\mathrm{p}_a^{(2)}(\boldsymbol{\Xi}, \boldsymbol{Q})$ satisfies condition (ii). Substitution of (2.33) into (2.32) yields

$$\mathrm{j}_k = \int e^{b_{k,2}+n^{-1/2}b_{k,4}+n^{-1}\mathrm{p}_c^{(1)}(\widehat{\boldsymbol{Q}},\Phi,\Theta)} d\boldsymbol{\varphi}_k^+ d\boldsymbol{\varphi}_k d\boldsymbol{\vartheta}_k^+ d\boldsymbol{\vartheta}_k + n^{-3/2}\mathrm{p}_a^{(2)}(\boldsymbol{\Xi}, \boldsymbol{Q}).$$

Further expansion implies

$$\mathrm{j}_k = \int \Bigl(1 + n^{-1/2}b_{k,4} + n^{-1}\mathrm{p}_c^{(2)}(\widehat{\boldsymbol{Q}}, \Phi, \Theta)\Bigr)\, e^{b_{k,2}} d\boldsymbol{\varphi}_k^+ d\boldsymbol{\varphi}_k d\boldsymbol{\vartheta}_k^+ d\boldsymbol{\vartheta}_k + n^{-3/2}\mathrm{p}_a^{(2)}(\boldsymbol{\Xi}, \boldsymbol{Q}),$$

where $\mathrm{p}_c^{(2)}(\widehat{\boldsymbol{Q}}, \Phi, \Theta)$ is again a polynomial such that $\mathrm{p}_c^{(2)}(0, \Phi, \Theta) = 0$. Similarly to the above, we obtain

$$\mathrm{j}_k = \int \Bigl(1 + n^{-1/2}b_{k,4}\Bigr)\, e^{b_{k,2}} d\boldsymbol{\varphi}_k^+ d\boldsymbol{\varphi}_k d\boldsymbol{\vartheta}_k^+ d\boldsymbol{\vartheta}_k + n^{-1}\mathrm{p}_c(\widehat{\boldsymbol{Q}}) + n^{-3/2}\mathrm{p}_a^{(2)}(\boldsymbol{\Xi}, \boldsymbol{Q}), \quad (2.34)$$

where $\mathrm{p}_c(\widehat{\boldsymbol{Q}})$ satisfies condition (i).

Recalling the definition of $y_{\alpha\beta}^{(k,p,s)}$ (2.23) and the values of $\kappa_{p,s}$ (2.20), one can render $b_{k,2}$ in the form

$$b_{k,2} = -\frac{1}{2}\boldsymbol{\rho}_k^T F \boldsymbol{\rho}_k, \quad (2.35)$$

where F is defined in (2.4) and

$$\boldsymbol{\rho}_k = \begin{pmatrix} (\boldsymbol{\varphi}_k^+)^T \\ (\boldsymbol{\vartheta}_k^+)^T \\ \boldsymbol{\varphi}_k \\ \boldsymbol{\vartheta}_k \end{pmatrix}. \tag{2.36}$$

Then (2.34) and (2.16) imply the assertion of the lemma. □

A substitution of (2.29) into (2.26) gives us

$$\mathsf{f}_m = \left(\frac{n}{\pi}\right)^{c_m} \int (h(\boldsymbol{Q}) + n^{-3/2}\mathrm{p}_a^{(2)}(\boldsymbol{\Xi}, \boldsymbol{Q}))^n \prod_{\substack{p+s \text{ is odd} \\ 0\le p,s\le m}} e^{-\operatorname{tr}\Xi_{p,s}^+\Xi_{p,s}} d\Xi_{p,s}^+ d\Xi_{p,s}$$
$$\times \prod_{\substack{p+s \text{ is even} \\ 0\le p,s\le m}} e^{-n\operatorname{tr}Q_{p,s}^*Q_{p,s}} dQ_{p,s}^* dQ_{p,s},$$

where $h(\boldsymbol{Q})$ is defined in (2.3). Further,

$$(h(\boldsymbol{Q}) + n^{-3/2}\mathrm{p}_a^{(2)}(\boldsymbol{\Xi}, \boldsymbol{Q}))^n = \sum_{k=0}^{c_m} \binom{n}{k} n^{-3k/2} h(\boldsymbol{Q})^{n-k} (\mathrm{p}_a^{(2)}(\boldsymbol{\Xi}, \boldsymbol{Q}))^k$$

because there are $2c_m$ anti-commuting variables and every monomial of $\mathrm{p}_a^{(2)}$ has at least second degree with respect to $\boldsymbol{\Xi}$. Hence,

$$\mathsf{f}_m = \left(\frac{n}{\pi}\right)^{c_m} \int (h(\boldsymbol{Q})^{c_m} + n^{-1/2}\mathrm{p}_a^{(3)}(\boldsymbol{\Xi}, \boldsymbol{Q}))$$
$$\times \prod_{\substack{p+s \text{ is odd} \\ 0\le p,s\le m}} e^{-\operatorname{tr}\Xi_{p,s}^+\Xi_{p,s}} d\Xi_{p,s}^+ d\Xi_{p,s} \times e^{nf(\boldsymbol{Q}) - c_m \log h(\boldsymbol{Q})} d\boldsymbol{Q}, \tag{2.37}$$

where $\mathrm{p}_a^{(3)}$ is a polynomial and $f(\boldsymbol{Q})$ is defined in (2.2). Taking into account (2.15) and the definition of an integral over anti-commuting variables, one can perform the integration over $\boldsymbol{\Xi}$ in (2.37) and obtain (2.1).

3. Asymptotic analysis

The goal of the section is to study the asymptotic behavior of the integral representation (2.6). To this end, the steepest descent method is applied. As usual, the hardest step is to choose the stationary points of $f(\boldsymbol{Q})$ and an N-dimensional (real) manifold $M_* \subset \mathbb{C}^N$ such that for any chosen stationary point $\boldsymbol{Q}_* \in M_*$

$$\Re f(\boldsymbol{Q}) < \Re f(\boldsymbol{Q}_*), \quad \forall \boldsymbol{Q} \in M_*,$$

where $\boldsymbol{Q}$ is not chosen. Note that N is equal to the number of real variables of the integration, i.e., in our case $N = 2^{2m}$.

The present proof proceeds by a standard scheme for the case when the function $f(\boldsymbol{Q})$ has the form

$$f(\boldsymbol{Q}) = f_0(\boldsymbol{Q}) + n^{-1/2} f_r(\boldsymbol{Q}),$$

where $f_0(\boldsymbol{Q})$ does not depend on n, whereas $f_r(\boldsymbol{Q})$ may depend on n. We choose the stationary points of $f_0(\boldsymbol{Q})$ of the form $\check{Q} = U\check{\Lambda}_0 U^T$, $\widehat{\boldsymbol{Q}} = 0$, where

$$\check{\Lambda}_0 = \begin{pmatrix} 0 & \Lambda_0 \\ -\Lambda_0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda_0 I_m \\ -\lambda_0 I_m & 0 \end{pmatrix},$$

λ_0 is a fixed positive number and U varies in $U(2m)$. Then the steepest descent method is applied to the integral over Λ and $\widehat{\boldsymbol{Q}}$. In the process U is considered as a parameter and all the estimates are uniform in U. As soon as the domain of integration is restricted by a small neighborhood, we recall about the integration over U. After several changes of the variables the integral is reduced to the form (1.9).

We start with the analysis of f_0.

Lemma 3.1. *Let the function* $f_0\colon \mathbb{R}^{2^{2m}} \to [-\infty, +\infty)$ *be defined by* (2.7). *Then* $f_0(\check{\Lambda}, \widehat{\boldsymbol{Q}})$ *attains its global maximum value only at the point*

$$\lambda_1 = \cdots = \lambda_m = \lambda_0, \quad \widehat{\boldsymbol{Q}} = 0,$$

where $\lambda_0 = \sqrt{1 - |z_0|^2}$. *Moreover, the matrix of the second order derivatives of* f_0 *w.r.t.* Λ *and* $\widehat{\boldsymbol{Q}}$ *at this point is negative definite.*

Proof. It is evident from (2.7) and (2.8) that $f_0(\check{\Lambda}, \widehat{\boldsymbol{Q}})$ has the form

$$f_0(\check{\Lambda}, \widehat{\boldsymbol{Q}}) = \sum_{j=1}^{m} f_*(\lambda_j) - \langle \boldsymbol{Q}_0, \boldsymbol{Q}_0 \rangle - \sum_{j=2}^{m} \langle \boldsymbol{Q}_j, \boldsymbol{Q}_j \rangle, \tag{3.1}$$

where

$$f_*(\lambda) = -\lambda^2 + \log(|z_0|^2 + \lambda^2).$$

Since $f_*'(\lambda) = 0$ iff $\lambda = \lambda_0$ and $\lim\limits_{\lambda\to\infty} f_*(\lambda) = -\infty$, $f_*(\lambda)$ attains its global maximum value only at $\lambda = \lambda_0$. Furthermore, $f_*''(\lambda_0) = -4\lambda_0^2$. These facts and (3.1) immediately imply the assertion of the lemma. □

To simplify the reading, the remaining steps are first explained in the case when the matrices M_n are from GinOE.

3.1. Gaussian case. Now we proceed to the integral estimates. In a standard way the integration domain in (2.11) can be restricted as follows:

$$\mathsf{f}_m = Cn^{2m^2-m} \int\limits_{\Sigma_r} \triangle^4(\Lambda^2) \prod_{j=1}^{m} \lambda_j \times e^{nf(U\check{\Lambda}U^T)} d\mu(U) d\Lambda + O(e^{-nr/2}),$$

where

$$\Sigma_r = \{(\Lambda, U) \mid \|\Lambda\| \le r\}.$$

The next step is to restrict the integration domain by

$$\Omega_n = \left\{(\Lambda, U) \mid \|\Lambda - \Lambda_0\| \le \frac{\log n}{\sqrt{n}}\right\}, \tag{3.2}$$

where $\Lambda_0 = \lambda_0 I_m$. To this end we need the estimate of $\Re f$ given by the following lemmas.

Lemma 3.2. *Let $\widetilde{\Lambda}$ be an $m \times m$ diagonal matrix such that $\|\widetilde{\Lambda}\| \le \log n$. Then, uniformly in U,*

$$\begin{aligned} f(U(\check{\Lambda}_0 + n^{-1/2}\check{\widetilde{\Lambda}})U^T) = &-m\lambda_0^2 + n^{-1/2} z_0 \operatorname{tr} \check{\mathcal{Z}} + n^{-1}\operatorname{tr}(\check{\mathcal{Z}}_U \check{\mathcal{Z}}_U^R)/2 \\ &- n^{-1}\operatorname{tr}(2\lambda_0\mathcal{M} + z_0\check{\mathcal{Z}}_U + z_0\check{\mathcal{Z}}_U^R)^2/4 \\ &+ O\big(n^{-3/2}\log^3 n\big), \end{aligned} \tag{3.3}$$

where $\check{\mathcal{Z}}$ is defined in (1.11), $\check{\mathcal{Z}}_W = W^*\check{\mathcal{Z}}W$, A^R is a dual matrix defined in (1.8), $\mathcal{M} = \operatorname{diag}\{\widetilde{\Lambda}, \widetilde{\Lambda}\}$.

Proof. If $\check{Q} = U(\check{\Lambda}_0 + n^{-1/2}\check{\widetilde{\Lambda}})U^T$, then $\check{F}$ has the form

$$\check{F} = \begin{pmatrix} U & 0 \\ 0 & \overline{U} \end{pmatrix} \left(\check{F}_0 + \frac{1}{\sqrt{n}}\check{F}_1\right) \begin{pmatrix} U^* & 0 \\ 0 & U^T \end{pmatrix},$$

where

$$\check{F}_0 = \begin{pmatrix} z_0 I_{2m} & \check{\Lambda}_0 \\ \check{\Lambda}_0 & z_0 I_{2m} \end{pmatrix}, \quad \check{F}_1 = \begin{pmatrix} \check{\mathcal{Z}}_U & \check{\widetilde{\Lambda}} \\ \check{\widetilde{\Lambda}} & \check{\mathcal{Z}}_U^T \end{pmatrix}. \tag{3.4}$$

Taking into account that

$$\det \check{F}_0 = \left[\det \begin{pmatrix} z_0 & \lambda_0 \\ -\lambda_0 & z_0 \end{pmatrix} \det \begin{pmatrix} z_0 & -\lambda_0 \\ \lambda_0 & z_0 \end{pmatrix}\right]^m = 1,$$

one gets

$$\begin{aligned} \log\det \check{F} &= \operatorname{tr}\log(1 + n^{-1/2}\check{F}_0^{-1}\check{F}_1) \\ &= \frac{1}{\sqrt{n}}\operatorname{tr}\check{F}_0^{-1}\check{F}_1 - \frac{1}{2n}\operatorname{tr}(\check{F}_0^{-1}\check{F}_1)^2 + O\left(\frac{\log^3 n}{\sqrt{n^3}}\right) \end{aligned} \tag{3.5}$$

uniformly in U. Moreover,

$$\check{F}_0^{-1}\check{F}_1 = \begin{pmatrix} z_0\check{\mathcal{Z}}_U + \lambda_0\mathcal{M} & z_0\check{\widetilde{\Lambda}} - \check{\Lambda}_0\check{\mathcal{Z}}_U^T \\ -\check{\Lambda}_0\check{\mathcal{Z}}_U + z_0\check{\widetilde{\Lambda}} & \lambda_0\mathcal{M} + z_0\check{\mathcal{Z}}_U^T \end{pmatrix}. \tag{3.6}$$

Combining (3.5), (3.6), (2.12) and the fact that $\check{\Lambda}_0\check{\widetilde{\Lambda}} = \check{\widetilde{\Lambda}}\check{\Lambda}_0 = -\lambda_0\mathcal{M}$, we get

$$f(U(\check{\Lambda}_0 + n^{-1/2}\check{\widetilde{\Lambda}})U^T)$$

$$= \frac{1}{2} \operatorname{tr} \Big[\check{\Lambda}_0^2 - 2n^{-1/2}\lambda_0 \mathcal{M} - n^{-1}\mathcal{M}^2 + n^{-1/2}(2\lambda_0\mathcal{M} + z_0\check{\mathcal{Z}}_U + z_0\check{\mathcal{Z}}_U^T)$$
$$- n^{-1}\big\{(\lambda_0^2 - z_0^2)\mathcal{M}^2 + 2z_0\lambda_0\check{\mathcal{Z}}_U\mathcal{M} + 2z_0\lambda_0\check{\mathcal{Z}}_U^T\mathcal{M}$$
$$+ \frac{1}{2}z_0^2\big(\check{\mathcal{Z}}_U^2 + (\check{\mathcal{Z}}_U^T)^2\big) + \check{\Lambda}_0\check{\mathcal{Z}}_U\check{\Lambda}_0\check{\mathcal{Z}}_U^T\big\}\Big] + O\big(n^{-3/2}\log^3 n\big). \qquad (3.7)$$

Note that

$$\operatorname{tr} \check{\mathcal{Z}}_U^T = \operatorname{tr} \check{\mathcal{Z}}_U = \operatorname{tr} \check{\mathcal{Z}},$$
$$\operatorname{tr} \check{\mathcal{Z}}_U^T \mathcal{M} = \operatorname{tr} \Big(\big(\check{\mathcal{Z}}_U^T\mathcal{M}\big)^T\Big)^R = \operatorname{tr} \check{\mathcal{Z}}_U^R \mathcal{M},$$
$$\operatorname{tr}\big(\check{\mathcal{Z}}_U^T\big)^2 = \operatorname{tr}\big(\check{\mathcal{Z}}_U^R\big)^2,$$
$$\operatorname{tr} \check{\Lambda}_0\check{\mathcal{Z}}_U\check{\Lambda}_0\check{\mathcal{Z}}_U^T = \lambda_0^2 \operatorname{tr} \check{\mathcal{Z}}_U J \check{\mathcal{Z}}_U^T J = -\lambda_0^2 \operatorname{tr} \check{\mathcal{Z}}_U\check{\mathcal{Z}}_U^R.$$

Hence the expansion (3.7) yields (3.3). □

Lemma 3.3. *Let $\widetilde{f}(\check{Q}) = f(\check{Q}) - f(\check{\Lambda}_0)$. Then, for sufficiently large n,*

$$\max_{\frac{\log n}{\sqrt{n}} \le \|\Lambda - \Lambda_0\| \le r} \Re \widetilde{f}(U\check{\Lambda}U^T) \le -C\frac{\log^2 n}{n}$$

uniformly in U.

Proof. First, let us check that the first and the second derivatives of f_r are bounded in the δ-neighborhood of Λ_0, where f_r is defined in (2.9) and δ is n-independent. Indeed, since h and h_0 are polynomials and $h \rightrightarrows h_0$ on compacts,

$$\left|\frac{1}{\sqrt{n}}\frac{\partial \Re f_r}{\partial \lambda_j}\right| \le \left|\frac{1}{\sqrt{n}}\frac{\partial f_r}{\partial \lambda_j}\right| = \left|\frac{\partial (f - f_0)}{\partial \lambda_j}\right| = \left|\frac{\partial(\log h - \log h_0)}{\partial \lambda_j}\right|$$
$$\le \left|\frac{1}{h_0}\frac{\partial h_0}{\partial \lambda_j} - \frac{1}{h}\frac{\partial h}{\partial \lambda_j}\right| \le \frac{C}{\sqrt{n}}.$$

For every real diagonal matrix E of unit norm and for $\frac{\log n}{\sqrt{n}} \le t \le \delta$, we have

$$\frac{d}{dt}\Re \widetilde{f}(U(\check{\Lambda}_0 + t\check{E})U^T) = \langle \nabla_\Lambda f_0(U(\check{\Lambda}_0 + t\check{E})U^T), v(E)\rangle$$
$$+ n^{-1/2}\langle \nabla_\Lambda \Re f_r(U(\check{\Lambda}_0 + t\check{E})U^T), v(E)\rangle$$
$$= \langle \nabla_\Lambda f_0(\check{\Lambda}_0 + t\check{E}), v(E)\rangle + O(n^{-1/2}),$$

where $\check{E} = \operatorname{diag}\{E, E\}$, $v(E)$ denotes a vector with components e_j and $\langle\cdot,\cdot\rangle$ is a standard real scalar product. Expanding the scalar product by the Taylor formula and considering that $\nabla_\Lambda f_0(\check{\Lambda}_0) = 0$, we obtain

$$\frac{d}{dt}\Re \widetilde{f}(U(\check{\Lambda}_0 + t\check{E})U^T) = t\langle f_0''(\check{\Lambda}_0)v(E), v(E)\rangle + r_1 + O(n^{-1/2}),$$

where f_0'' is a matrix of second order derivatives of f_0 with respect to Λ and $|r_1| \le Ct^2$. $f_0''(\check\Lambda_0)$ is negative definite according to Lemma 3.1. Hence $\frac{d}{dt}\Re\widetilde{f}(U(\check\Lambda_0 + t\check{E})U^T)$ is negative and

$$\max_{\frac{\log n}{\sqrt{n}} \le \|\Lambda-\Lambda_0\| \le \delta} \Re\widetilde{f}(U\check\Lambda U^T) = \max_{\|\Lambda-\Lambda_0\| = \frac{\log n}{\sqrt{n}}} \Re\widetilde{f}(U\check\Lambda U^T)$$
$$\le f(U\check\Lambda_0 U^T) - C\frac{\log^2 n}{n} - f(\check\Lambda_0). \tag{3.8}$$

Notice that f_r is bounded from above uniformly in n. This fact and Lemma 3.1 imply that δ in (3.8) can be replaced by r,

$$\max_{\frac{\log n}{\sqrt{n}} \le \|\Lambda-\Lambda_0\| \le r} \Re\widetilde{f}(U\check\Lambda U^T) \le f(U\check\Lambda_0 U^T) - f(\check\Lambda_0) - C\frac{\log^2 n}{n}.$$

It remains to deduce from Lemma 3.2 that $f(U\check\Lambda_0 U^T) - f(\Lambda_0) = O(n^{-1})$ uniformly in U. □

Lemma 3.3 yields

$$\mathsf{f}_m = Cn^{2m^2-m}e^{nf(\check\Lambda_0)}\left(\int_{\Omega_n} \triangle^4(\Lambda^2)\prod_{j=1}^m \lambda_j \times e^{n\widetilde{f}(U\check\Lambda U^T)}d\mu(U)d\Lambda + O(e^{-C_1\log^2 n})\right),$$

where Ω_n is defined in (3.2). Changing the variables $\Lambda = \Lambda_0 + \frac{1}{\sqrt{n}}\widetilde\Lambda$ and expanding f according to Lemma 3.2, we obtain

$$\mathsf{f}_m = C\mathsf{k}_n \int_{\sqrt{n}\Omega_n} \triangle^4(\widetilde\Lambda)d\widetilde\Lambda d\mu(U)(1+o(1)) \tag{3.9}$$
$$\times \exp\left\{-\frac{1}{4}\operatorname{tr}(2\lambda_0\mathcal{M} + z_0\check{\mathcal{Z}}_U + z_0\check{\mathcal{Z}}_U^R)^2 + \frac{1}{2}\operatorname{tr}\check{\mathcal{Z}}_U\check{\mathcal{Z}}_U^R\right\}, \tag{3.10}$$

where

$$\mathsf{k}_n = n^{m^2-m/2}e^{-mn\lambda_0^2+\sqrt{n}z_0\operatorname{tr}\check{\mathcal{Z}}}. \tag{3.11}$$

Let us do several changes of the variables. The first one is $U = O\check{D}S^*$, where O is a real orthogonal matrix, S is a unitary symplectic matrix and $\check{D} = \operatorname{diag}\{D, D\}$, $D = \operatorname{diag}\{e^{i\eta_j}\}_{j=1}^m$. Taking into account that $d\mu(U)$ changes to $Cd\mu(O)d\mu(S)d\eta$ with $d\eta = \triangle^2(D^4)\prod_{j=1}^m e^{-(4m-4)i\eta_j}d\eta_j$, we get

$$\mathsf{f}_m = C\mathsf{k}_n\int_{\mathbb{R}^m}\triangle^4(\widetilde\Lambda)d\widetilde\Lambda \int_{[0,\pi]^m} d\eta \int_{O(2m)} d\mu(O) \int_{\mathrm{USp}(m)} d\mu(S)(1+o(1))$$
$$\times \exp\left\{-\frac{1}{4}\operatorname{tr}(2\lambda_0\mathcal{M} + S^*(z_0\check{\mathcal{Z}}_{O\check{D}} + z_0\check{\mathcal{Z}}_{O\check{D}}^R)S)^2\right.$$
$$\left. + \frac{1}{2}\operatorname{tr}\check{\mathcal{Z}}O\check{D}^2O^R\check{\mathcal{Z}}^R\left(O\check{D}^2O^R\right)^*\right\}$$

$$= C\mathsf{k}_n \int_{\mathbb{R}^m} \triangle^4(\widetilde{\Lambda})d\widetilde{\Lambda} \int_{[0,\pi]^m} d\eta \int_{O(2m)} d\mu(O) \int_{\mathrm{USp}(m)} d\mu(S)(1+o(1))$$
$$\times \exp\left\{-\frac{1}{4}\operatorname{tr}(2\lambda_0 S\mathcal{M}S^* + z_0\check{Z}_{O\check{D}} + z_0\check{Z}^R_{O\check{D}})^2 + \frac{1}{2}\operatorname{tr}\check{Z}V\check{Z}^RV^*\right\},$$

where $V = O\check{D}^2O^R$. The second change of the variables is $H = S\mathcal{M}S^*$. Then H runs over Hermitian self-dual matrices, and the Jacobian of the change is $C\triangle^{-4}(\widetilde{\Lambda})$. Thus,

$$\mathsf{f}_m = C\mathsf{k}_n \int_{H=H^*=H^R} dH \int_{[0,\pi]^m} d\eta \int_{O(2m)} d\mu(O)(1+o(1))$$
$$\times \exp\left\{-\frac{1}{4}\operatorname{tr}(2\lambda_0 H + z_0\check{Z}_{O\check{D}} + z_0\check{Z}^R_{O\check{D}})^2 + \frac{1}{2}\operatorname{tr}\check{Z}V\check{Z}^RV^*\right\},$$

where

$$dH = \prod_{j=1}^m d(H)_{jj} \prod_{j<k\le m} d\Re(H)_{jk}d\Im(H)_{jk}d\Re(H)_{j,k+m}d\Im(H)_{j,k+m}.$$

The Gaussian integration over H implies

$$\mathsf{f}_m = C\mathsf{k}_n \int_{[0,\pi]^m} d\eta \int_{O(2m)} d\mu(O)\exp\left\{\frac{1}{2}\operatorname{tr}\check{Z}V\check{Z}^RV^*\right\}(1+o(1)). \tag{3.12}$$

Finally, the last change of the variables $V = O\check{D}^2O^R$ brings the integration domain to the set of all unitary self-dual $2m\times 2m$ matrices. $d\eta d\mu(O)$ transforms to the measure $Cd\mu_s(V)$ which corresponds to the differential form (1.10). Therefore,

$$\mathsf{f}_m = C\mathsf{k}_n \int_{V=V^R\in U(2m)} \exp\left\{\frac{1}{2}\operatorname{tr}\check{Z}V\check{Z}^RV^*\right\}d\mu_s(V)(1+o(1)). \tag{3.13}$$

(3.13) and (2.10) yield assertion (i) of Theorem 1.1.

In order to prove assertion (ii), let us compute the integral (3.13) for $m = 2$.

Lemma 3.4. *Let $A = \operatorname{diag}\{a_1,a_2,a_3,a_4\}$ be an arbitrary diagonal matrix. Then*

$$\int_{V=V^R\in U(4)} \exp\left\{\frac{1}{2}\operatorname{tr}AVA^RV^*\right\}d\mu_s(V) = C\frac{\operatorname{Pf}((a_j-a_k)e^{a_ja_k})^4_{j,k=1}}{\triangle(A)}. \tag{3.14}$$

Proof. Observe that the left-hand side of (3.14) is analytic in a_1, a_2, a_3, a_4. Thus, it is sufficient to evaluate series of the integral at $A = 0$. A straightforward computation gives us

$$\frac{1}{2}\operatorname{tr}AVA^RV^* = a_1a_3 + a_2a_4 - (a_1-a_2)(a_3-a_4)\left|v_{12}\right|^2 - (a_2-a_3)(a_4-a_1)\left|v_{14}\right|^2.$$

Let us define the number sequence $\{\mathsf{c}_{jk}\}$ by the equality

$$e^{-b_0} \int\limits_{V=V^R\in U(4)} \exp\left\{\frac{1}{2}\operatorname{tr} AVA^RV^*\right\} d\mu_s(V) = \sum_{j,k=0}^{\infty} \mathsf{c}_{jk} b_1^j b_2^k, \tag{3.15}$$

where

$$\begin{aligned} b_0 &= a_1a_3 + a_2a_4, \\ b_1 &= (a_1 - a_2)(a_3 - a_4), \\ b_2 &= (a_2 - a_3)(a_4 - a_1). \end{aligned} \tag{3.16}$$

Further,

$$\begin{aligned} \mathsf{c}_{jk} &= \int\limits_{V=V^R\in U(4)} \frac{(-1)^{j+k}}{j!k!} |v_{12}|^{2j} |v_{14}|^{2k} d\mu_s(V) \\ &= \int\limits_{V=V^R\in U(4)} \frac{(-1)^k}{j!k!} \times \frac{v_{12}^j v_{21}^j}{\operatorname{Pf}^j VJ} \times \frac{v_{14}^k v_{32}^k}{\operatorname{Pf}^k VJ} d\mu_s(V). \end{aligned} \tag{3.17}$$

In order to compute the last integral, the following bosonization formula (see [27, Theorem 4.11]) is used:

$$\begin{aligned} &\int \mathtt{f}\left(\begin{pmatrix} \Upsilon^+\Upsilon & \Upsilon^+\left(\Upsilon^+\right)^T \\ -\left(\Upsilon\right)^T\Upsilon & -\left(\Upsilon\right)^T\left(\Upsilon^+\right)^T \end{pmatrix}\right) d\Upsilon^+ d\Upsilon \\ &\qquad = (2\pi)^{qn} 2^q \frac{\operatorname{vol}(O_n)}{\operatorname{vol}(O_{n+2q})} \int\limits_{V=V^R\in U(2q)} \mathtt{f}(V) \det{}^{-n/2} V d\mu_s(V), \end{aligned} \tag{3.18}$$

where Υ is an $n\times q$ matrix with anti-commuting entries, $\mathtt{f}$ is an analytic function and vol stands for volume. Let us apply (3.18) to (3.17) for $n = j+k$ and $q = 2$. Taking into account that $\operatorname{vol}(O_n) = \frac{2^n \pi^{\frac{n(n+1)}{4}}}{\prod_{p=1}^n \Gamma(p/2)}$, we obtain

$$\begin{aligned} \mathsf{c}_{jk} = C &\frac{(-1)^k}{j!k!(j+k+2)!(j+k)!} \\ &\times \int \left(\sum_{l=1}^{j+k} v_{l1}^* v_{l2}\right)^j \left(\sum_{l=1}^{j+k} v_{l2}^* v_{l1}\right)^j \left(\sum_{l=1}^{j+k} v_{l1}^* v_{l2}^*\right)^k \left(-\sum_{l=1}^{j+k} v_{l1} v_{l2}\right)^k d\Upsilon^+ d\Upsilon, \end{aligned}$$

where C depends neither on j nor on k. Doing some combinatorics, it is easy to see that the integrand equals

$$(-1)^j (j+k)! j! k! \prod_{l=1}^{j+k} v_{l1}^* v_{l1} v_{l2}^* v_{l2}.$$

Hence,

$$\mathsf{c}_{jk} = C \frac{(-1)^{j+k}}{(j+k+2)!}.$$

Summing over j and k, we get

$$\sum_{j,k=0}^{\infty} \mathsf{c}_{jk} b_1^j b_2^k = C \sum_{l=0}^{\infty} \frac{(-1)^l}{(l+2)!} \sum_{j+k=l} b_1^j b_2^k$$
$$= C \sum_{l=0}^{\infty} \frac{(-1)^l}{(l+2)!} \frac{b_1^{l+1} - b_2^{l+1}}{b_1 - b_2} = C \frac{\frac{e^{-b_1}-1}{b_1} - \frac{e^{-b_2}-1}{b_2}}{b_1 - b_2}. \tag{3.19}$$

Note that $b_1 - b_2 = (a_1 - a_3)(a_2 - a_4)$. Thus, (3.15), (3.16) and (3.19) imply

$$\int_{\substack{V=V^R\\ V\in U(4)}} \exp\left\{\frac{1}{2}\operatorname{tr} AVA^RV^*\right\} d\mu_s(V) = Ce^{b_0} \frac{b_2 e^{-b_1} - b_1 e^{-b_2} + (b_1 - b_2)}{b_1 b_2 (b_1 - b_2)}$$
$$= \frac{C}{-\triangle(A)} \big[(a_2 - a_3)(a_4 - a_1)e^{a_1a_4 + a_2a_3} - (a_1 - a_2)(a_3 - a_4)e^{a_1a_2 + a_3a_4} + (a_1 - a_3)(a_2 - a_4)e^{a_1a_3 + a_2a_4}\big].$$

To finish the proof, it remains to observe that the expression in the brackets is exactly $-\operatorname{Pf}((a_j - a_k)e^{a_ja_k})_{j,k=1}^4$. □

Applying the lemma to (3.13), one obtains

$$\mathsf{f}_2 = C\mathsf{k}_n \frac{\operatorname{Pf}\begin{pmatrix} (\zeta_j - \zeta_k)e^{\zeta_j\zeta_k} & (\zeta_j - \bar\zeta_k)e^{\zeta_j\bar\zeta_k} \\ (\bar\zeta_j - \zeta_k)e^{\bar\zeta_j\zeta_k} & (\bar\zeta_j - \bar\zeta_k)e^{\bar\zeta_j\bar\zeta_k}\end{pmatrix}_{j,k=1}^2}{\triangle(\zeta_1, \zeta_2, \bar\zeta_1, \bar\zeta_2)}(1 + o(1)),$$

which in combination with (2.10) yields assertion (ii) of Theorem 1.1.

3.2. General case. In the general case the proof proceeds by the same scheme as in the Gaussian case. In this subsection, we focus on the crucial distinctions from the Gaussian case and refine the corresponding assertions from the previous subsection.

In order to formulate the refinement of Lemma 3.2, let us introduce some new notations. Set

$$\|\widehat{\boldsymbol{Q}}\| = \sqrt{\langle \widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{Q}}\rangle}.$$

Since x_{11} is real, $\kappa_{p,s}$ depends only on $p+s$ (see (2.19)). We denote the common value of $\kappa_{4-s,s}$ by κ_4. It is also convenient to change indices of the entries of $\boldsymbol{Q}_2$. For any $\delta \in \mathcal{I}_{2m,4}$, determine a number s such that $\delta_s \le m < \delta_{s+1}$. Then put $\widetilde{q}_\delta^{(2)} = \widetilde{q}_{\alpha\beta}^{(4-s,s)}$, where

$$\alpha = (\delta_{s+1} - m, \dots, \delta_4 - m);$$
$$\beta = (\delta_1, \dots, \delta_s).$$

$\mathcal{I}'_{2m,4}$ denotes a set of such indices δ that $s = 2$ and $\alpha = \beta$. $\wedge^4 B$ is the fourth exterior power of a linear operator B (see [41] for definition and properties of an exterior power of a linear operator).

At the point we are ready to generalize Lemma 3.2.

Lemma 3.5. *Let* $\|\widetilde{\Lambda}\| + \|\widehat{\widetilde{\boldsymbol{Q}}}\| \le \log n$. *Then, uniformly in* U *and* V,

$$\begin{aligned} f(U(\check{\Lambda}_0 + n^{-1/2}\check{\widetilde{\Lambda}})U^T, n^{-1/2}\widehat{\widetilde{\boldsymbol{Q}}}) &= -m\lambda_0^2 + n^{-1/2} z_0 \operatorname{tr} \check{\mathcal{Z}} + n^{-1} \operatorname{tr}(\check{\mathcal{Z}}_U \check{\mathcal{Z}}_U^R)/2 \\ &\quad - n^{-1} \operatorname{tr}(2\lambda_0 \mathcal{M} + z_0 \check{\mathcal{Z}}_U + z_0 \check{\mathcal{Z}}_U^R)^2/4 \\ &\quad + n^{-1}\lambda_0^2 \sqrt{\kappa_4} \sum_{\delta \in \mathcal{I}_{2m,4}} \sum_{\gamma \in \mathcal{I}'_{2m,4}} ((\wedge^4 \overline{U})_{\delta\gamma} \widetilde{q}_\delta^{(2)} + \bar{\widetilde{q}}_\delta^{(2)} (\wedge^4 U^T)_{\gamma\delta}) \\ &\quad - n^{-1} \langle \widehat{\widetilde{\boldsymbol{Q}}}, \widehat{\widetilde{\boldsymbol{Q}}} \rangle + O\big(n^{-3/2} \log^3 n\big), \end{aligned} \tag{3.20}$$

where we keep the notations of Lemma 3.2. All new notations are described just before this lemma.

Proof. Differently from the Gaussian case, f has an additional term $\langle \widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{Q}} \rangle$, and an additional term $n^{-1/2}\widetilde{h}(\boldsymbol{Q}_2) + n^{-1}\mathrm{p}_c(\widehat{\boldsymbol{Q}})$ under the logarithm (cf. (2.2) and (2.12)), where $\widetilde{h}$ and p_c are defined in the assertion of Lemma 2.6. The contribution of the term $\langle \widehat{\boldsymbol{Q}}, \widehat{\boldsymbol{Q}} \rangle$ to the expansion (3.20) is evident. Furthermore, $n^{-1}\mathrm{p}_c(n^{-1/2}\widehat{\widetilde{\boldsymbol{Q}}}) = O\big(n^{-3/2}\log^3 n\big)$ because p_c is a polynomial with zero constant term. Hence, it remains to determine the contribution of the term $n^{-1/2}\widetilde{h}(\boldsymbol{Q}_2)$.

In order to simplify notations, let us omit the index k in (2.30). Thus, now $\boldsymbol{\varphi}$ and $\boldsymbol{\vartheta}$ denote the vectors

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_m \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_m \end{pmatrix},$$

respectively. Then (2.30) is written as

$$\widetilde{h}(\boldsymbol{Q}_2) = \int b_4 e^{b_2} d\boldsymbol{\varphi}^+ d\boldsymbol{\varphi} d\boldsymbol{\vartheta}^+ d\boldsymbol{\vartheta},$$

where b_2 has the form (2.35) and b_4 is defined in (2.28). Therefore,

$$\begin{aligned} n^{-1/2}\widetilde{h}(n^{-1/2}\widetilde{\boldsymbol{Q}}_2) = n^{-1}\widetilde{h}(\widetilde{\boldsymbol{Q}}_2) = -\frac{1}{n}\int d\boldsymbol{\varphi}^+ d\boldsymbol{\varphi} d\boldsymbol{\vartheta}^+ d\boldsymbol{\vartheta}\, e^{-\frac{1}{2}\boldsymbol{\rho}^T F \boldsymbol{\rho}} \\ \times \sum_{p+s=4} \Big(\operatorname{tr} \widetilde{Y}_{p,s}\widetilde{Q}_{p,s} + \operatorname{tr} \widetilde{Q}_{p,s}^* Y_{p,s}\Big), \end{aligned} \tag{3.21}$$

where $\boldsymbol{\rho}$ is defined in (2.36), F is defined in (2.4), $\widetilde{Y}_{p,s}$ and $Y_{p,s}$ are defined by (2.23). In order to make further computation more clear, we introduce some more notations. Put

$$\boldsymbol{\rho}_1 = \begin{pmatrix} \boldsymbol{\vartheta} \\ (\boldsymbol{\varphi}^+)^T \end{pmatrix}, \qquad \boldsymbol{\rho}_1^+ = \begin{pmatrix} \boldsymbol{\vartheta}^+ & -\boldsymbol{\varphi}^T \end{pmatrix}. \tag{3.22}$$

Let us change the variables $\widetilde{\boldsymbol{\rho}}_1 = U^T \boldsymbol{\rho}_1$, $\widetilde{\boldsymbol{\rho}}_1^+ = \boldsymbol{\rho}_1^+ \overline{U}$. We have

$$\kappa_4^{-1/2} y_{\alpha\beta}^{(p,s)} = \prod_{q=1}^{p} \phi_{\alpha_q} \prod_{r=1}^{s} \theta_{\beta_r}^* = (-1)^p \prod_{q=1}^{p} \left(\boldsymbol{\rho}_1^+\right)_{m+\alpha_q} \prod_{r=1}^{s} \left(\boldsymbol{\rho}_1^+\right)_{\beta_r}$$
$$= (-1)^p (-1)^{\frac{p(p-1)}{2} + \frac{s(s-1)}{2}} \prod_{r=p+s}^{1} \left(\widetilde{\boldsymbol{\rho}}_1^+ U^T\right)_{\delta_r},$$

where

$$\delta = (\beta_1, \ldots, \beta_s, m + \alpha_1, \ldots, m + \alpha_p) \in \mathcal{I}_{2m,4}.$$

Taking into account that $p + s = 4$ and

$$p + \frac{p(p-1)}{2} + \frac{s(s-1)}{2} = p(p-3) + 6$$

is even, we get

$$\kappa_4^{-1/2} y_{\alpha\beta}^{(p,s)} = \sum_{\gamma \in ([1,2m]\cap\mathbb{Z})^4} \prod_{r=4}^{1} \left(\widetilde{\boldsymbol{\rho}}_1^+\right)_{\gamma_r} u_{\delta_r\gamma_r}$$
$$= \sum_{\gamma \in \mathcal{I}_{2m,4}} \det \left\{U^T\right\}_{\gamma\delta} \prod_{r=4}^{1} \left(\widetilde{\boldsymbol{\rho}}_1^+\right)_{\gamma_r}$$
$$= \sum_{\gamma \in \mathcal{I}_{2m,4}} (\wedge^4 U^T)_{\gamma\delta} \prod_{r=4}^{1} \left(\widetilde{\boldsymbol{\rho}}_1^+\right)_{\gamma_r}, \tag{3.23}$$

where $u_{jk} = (U)_{jk}$ and $\left\{U^T\right\}_{\gamma\delta}$ is a submatrix of U^T constructed as an intersection of rows $\gamma_1, \ldots, \gamma_4$ with columns $\delta_1, \ldots, \delta_4$. Similarly,

$$\kappa_4^{-1/2} \widetilde{y}_{\beta\alpha}^{(p,s)} = \sum_{\gamma \in \mathcal{I}_{2m,4}} (\wedge^4 \overline{U})_{\delta\gamma} \prod_{r=1}^{4} (\widetilde{\boldsymbol{\rho}}_1)_{\gamma_r}. \tag{3.24}$$

Besides,

$$\boldsymbol{\rho}^T F \boldsymbol{\rho} = -\boldsymbol{\rho}_*^+ \check{F} \boldsymbol{\rho}_* = -\widetilde{\boldsymbol{\rho}}_*^+ \check{F}_0 \widetilde{\boldsymbol{\rho}}_* + O(n^{-1/2} \log n), \tag{3.25}$$

where $\check{F}_0$ is defined in (3.4) and

$$\boldsymbol{\rho}_* = \begin{pmatrix} \left(\boldsymbol{\rho}_1^+\right)^T \\ \boldsymbol{\rho}_1 \end{pmatrix}, \quad \boldsymbol{\rho}_*^+ = \begin{pmatrix} -\boldsymbol{\rho}_1^T & \boldsymbol{\rho}_1^+ \end{pmatrix}, \quad \widetilde{\boldsymbol{\rho}} = \begin{pmatrix} \left(\widetilde{\boldsymbol{\rho}}_1^+\right)^T \\ \widetilde{\boldsymbol{\rho}}_1 \end{pmatrix}, \quad \widetilde{\boldsymbol{\rho}}_*^+ = \begin{pmatrix} -\widetilde{\boldsymbol{\rho}}_1^T & \widetilde{\boldsymbol{\rho}}_1^+ \end{pmatrix}.$$

The "measure" changes as follows:

$$d\boldsymbol{\varphi}^+ d\boldsymbol{\varphi} d\boldsymbol{\vartheta}^+ d\boldsymbol{\vartheta} = \det{}^{-1} U^T \det{}^{-1} \overline{U} d\widetilde{\boldsymbol{\rho}}_1^+ d\widetilde{\boldsymbol{\rho}}_1 = d\widetilde{\boldsymbol{\rho}}_1^+ d\widetilde{\boldsymbol{\rho}}_1. \tag{3.26}$$

Eventually, substitution of (3.23)–(3.26) into (3.21) yields

$$n^{-1} \widetilde{h}(\widetilde{\boldsymbol{Q}}_2) = -\frac{1}{n} \sum_{\delta \in \mathcal{I}_{2m,4}} \sum_{\gamma \in \mathcal{I}_{2m,4}} \sqrt{\kappa_4} \int e^{\frac{1}{2} \widetilde{\boldsymbol{\rho}}_*^+ \check{F}_0 \widetilde{\boldsymbol{\rho}}_*} d\widetilde{\boldsymbol{\rho}}_1^+ d\widetilde{\boldsymbol{\rho}}_1$$

$$\times\Big((\wedge^4\overline{U})_{\delta\gamma}\prod_{r=1}^{4}(\widetilde{\boldsymbol{\rho}}_1)_{\gamma_r}\,\widetilde{q}_\delta^{(2)}+\bar{\widetilde{q}}_\delta^{(2)}(\wedge^4U^T)_{\gamma\delta}\prod_{r=4}^{1}(\widetilde{\boldsymbol{\rho}}_1^+)_{\gamma_r}\Big)$$
$$+O\big(n^{-3/2}\log^3 n\big) \tag{3.27}$$

uniformly in U.

Let us denote the components of $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_1^+$ in the same way as in (3.22) but with tildes. Then the integration in (3.27) can be performed over $\widetilde{\phi}_j$, $\widetilde{\theta}_j$ separately for every j due to the structure of $\check{F}_0$. Thus, it remains to compute the integral

$$\int\prod_{r=1}^{4}(\widetilde{\boldsymbol{\rho}}_1)_{\gamma_r}\exp\Big\{z_0\widetilde{\theta}_j\widetilde{\theta}_j^*+\lambda_0\widetilde{\theta}_j\widetilde{\phi}_j^*+z_0\widetilde{\phi}_j\widetilde{\phi}_j^*-\lambda_0\widetilde{\phi}_j\widetilde{\theta}_j^*\Big\}\,d\widetilde{\phi}_j^*d\widetilde{\phi}_jd\widetilde{\theta}_j^*d\widetilde{\theta}_j$$

and the same one with $\boldsymbol{\rho}_1^+$ instead of $\boldsymbol{\rho}_1$. Furthermore, expanding the exponent into series, one can observe that the integral is non-zero only if $\gamma\in\mathcal{I}'_{2m,4}$. Moreover,

$$\int\widetilde{\theta}_j\widetilde{\phi}_j^*e^{z_0\widetilde{\theta}_j\widetilde{\theta}_j^*+\lambda_0\widetilde{\theta}_j\widetilde{\phi}_j^*+z_0\widetilde{\phi}_j\widetilde{\phi}_j^*-\lambda_0\widetilde{\phi}_j\widetilde{\theta}_j^*}d\widetilde{\phi}_j^*d\widetilde{\phi}_jd\widetilde{\theta}_j^*d\widetilde{\theta}_j=\lambda_0,$$
$$\int\widetilde{\phi}_j\widetilde{\theta}_j^*e^{z_0\widetilde{\theta}_j\widetilde{\theta}_j^*+\lambda_0\widetilde{\theta}_j\widetilde{\phi}_j^*+z_0\widetilde{\phi}_j\widetilde{\phi}_j^*-\lambda_0\widetilde{\phi}_j\widetilde{\theta}_j^*}d\widetilde{\phi}_j^*d\widetilde{\phi}_jd\widetilde{\theta}_j^*d\widetilde{\theta}_j=-\lambda_0,$$
$$\int e^{z_0\widetilde{\theta}_j\widetilde{\theta}_j^*+\lambda_0\widetilde{\theta}_j\widetilde{\phi}_j^*+z_0\widetilde{\phi}_j\widetilde{\phi}_j^*-\lambda_0\widetilde{\phi}_j\widetilde{\theta}_j^*}d\widetilde{\phi}_j^*d\widetilde{\phi}_jd\widetilde{\theta}_j^*d\widetilde{\theta}_j=1.$$

It implies

$$n^{-1}\widetilde{h}(\widetilde{\boldsymbol{Q}}_2)=n^{-1}\lambda_0^2\sqrt{\kappa_4}\sum_{\delta\in\mathcal{I}_{2m,4}}\sum_{\gamma\in\mathcal{I}'_{2m,4}}((\wedge^4\overline{U})_{\delta\gamma}\widetilde{q}_\delta^{(2)}+\bar{\widetilde{q}}_\delta^{(2)}(\wedge^4U^T)_{\gamma\delta})$$
$$+O\big(n^{-3/2}\log^3 n\big).$$

The above relation completes the proof of (3.20). □

An analog of Lemma 3.3 is

Lemma 3.6. *Let $\widetilde{f}(\boldsymbol{Q})=f(\boldsymbol{Q})-f(\check{\Lambda}_0,0)$. Then, for sufficiently large n,*

$$\max_{\frac{\log n}{\sqrt{n}}\le\|\check{\Lambda}-\check{\Lambda}_0\|+\|\widehat{\boldsymbol{Q}}\|\le r}\Re\widetilde{f}(U\check{\Lambda}U^T,\widehat{\boldsymbol{Q}})\le -C\frac{\log^2 n}{n}$$

uniformly in U.

The proof needs only cosmetic changes because of additional variables $\widehat{\boldsymbol{Q}}$. Following the proof in the Gaussian case, one can see that (3.9) transforms into

$$\mathsf{f}_m=C\mathsf{k}_n\int_{\sqrt{n}\Omega_n}\triangle^4(\widetilde{\Lambda})\exp\Big\{-\frac{1}{4}\operatorname{tr}(2\lambda_0\mathcal{M}+z_0\check{\mathcal{Z}}_U+z_0\check{\mathcal{Z}}_U^R)^2+\frac{1}{2}\operatorname{tr}\check{\mathcal{Z}}_U\check{\mathcal{Z}}_U^R$$

$$+ \lambda_0^2\sqrt{\kappa_4} \sum_{\delta\in\mathcal{I}_{2m,4}} \sum_{\gamma\in\mathcal{I}'_{2m,4}} ((\wedge^4\overline{U})_{\delta\gamma}\widetilde{q}_\delta^{(2)} + \bar{\widetilde{q}}_\delta^{(2)}(\wedge^4 U^T)_{\gamma\delta})$$
$$- \langle \widehat{\widetilde{\boldsymbol{Q}}}, \widehat{\widetilde{\boldsymbol{Q}}}\rangle \Big\} d\mu(U)d\widetilde{\Lambda}d\widehat{\widetilde{\boldsymbol{Q}}}(1+o(1)),$$

where k_n is defined in (3.11). The Gaussian integration over $\widehat{\widetilde{\boldsymbol{Q}}}$ yields

$$\mathsf{f}_m = C\mathsf{k}_n \int \triangle^4(\widetilde{\Lambda}) \exp\Big\{-\frac{1}{4}\operatorname{tr}(2\lambda_0\mathcal{M} + z_0\check{\mathcal{Z}}_U + z_0\check{\mathcal{Z}}_U^R)^2 + \frac{1}{2}\operatorname{tr}\check{\mathcal{Z}}_U\check{\mathcal{Z}}_U^R$$
$$+ \lambda_0^4\kappa_4 \sum_{\delta\in\mathcal{I}_{2m,4}} \sum_{\gamma\in\mathcal{I}'_{2m,4}} (\wedge^4 U^T)_{\gamma\delta}(\wedge^4\overline{U})_{\delta\gamma}\Big\} d\mu(U)d\widetilde{\Lambda}(1+o(1)).$$

Note that $\wedge^4 U^T$ and $\wedge^4\overline{U}$ are mutually inverse matrices. Hence,

$$\sum_{\delta\in\mathcal{I}_{2m,4}} (\wedge^4 U^T)_{\gamma\delta}(\wedge^4\overline{U})_{\delta\gamma} = (\wedge^4 U^T \wedge^4 \overline{U})_{\gamma\gamma} = 1.$$

Therefore,

$$\mathsf{f}_m = C\mathsf{k}_n \exp\left\{\frac{m^2-m}{2}\lambda_0^4\kappa_4\right\}(1+o(1))$$
$$\times \int \triangle^4(\widetilde{\Lambda}) \exp\Big\{-\frac{1}{4}\operatorname{tr}(2\lambda_0\mathcal{M} + z_0\check{\mathcal{Z}}_U + z_0\check{\mathcal{Z}}_U^R)^2 + \frac{1}{2}\operatorname{tr}\check{\mathcal{Z}}_U\check{\mathcal{Z}}_U^R\Big\} d\mu(U)d\widetilde{\Lambda}.$$

The last formula shows that there are no differences in further proof up to a high moments independent factor $\exp\left\{\frac{m^2-m}{2}\lambda_0^4\kappa_4\right\}$.

Acknowledgments. The author is grateful to Prof. M. Shcherbina for the statement of the problem and fruitful discussions. The author is supported in part by the Akhiezer Foundation scholarship.

References

[1] I. Afanasiev, *On the correlation functions of the characteristic polynomials of the sparse Hermitian random matrices,* J. Stat. Phys. **163(2)** (2016), 324–356.

[2] I. Afanasiev, *On the correlation functions of the characteristic polynomials of non-Hermitian random matrices with independent entries,* J. Stat. Phys. **176(6)** (2019), 1561–1582.

[3] G. Akemann and E. Kanzieper, *Integrable structure of Ginibre's ensemble of real random matrices and a Pfaffian integration theorem,* J. Stat. Phys. **129(5-6)** (2007), 1159–1231.

[4] F.A. Berezin, *Introduction to Superanalysis*, Number 9 in Math. Phys. Appl. Math. D. Reidel Publishing Co., Dordrecht, 1987. Edited and with a foreword by A.A. Kirillov. With an appendix by V.I. Ogievetsky. Translated from the Russian by J. Niederle and R. Kotecký. Translation edited by D. Leĭtes.

[5] C. Bordenave and D. Chafaï, *Around the circular law,* Probab. Surv. **9** (2012), 1–89.

[6] A. Borodin and C. D. Sinclair, *The Ginibre ensemble of real random matrices and its scaling limits,* Comm. Math. Phys. **291** (2009), 177–224.

[7] A. Borodin and E. Strahov, *Averages of characteristic polynomials in random matrix theory,* Comm. Pure Appl. Math. **59(2)** (2006), 161–253.

[8] E. Brézin and S. Hikami, *Characteristic polynomials of random matrices,* Comm. Math. Phys. **214** (2000), 111–135.

[9] E. Brézin and S. Hikami. *Characteristic polynomials of real symmetric random matrices,* Comm. Math. Phys. **223** (2001), 363–382.

[10] G. Cipolloni, L. Erdős, and D. Schröder, *Edge universality for non-Hermitian random matrices,* preprint, https://arxiv.org/abs/1908.00969v2.

[11] G. Cipolloni, L. Erdős, and D. Schröder, *Optimal lower bound on the least singular value of the shifted Ginibre ensemble,* preprint, https://arxiv.org/abs/1908.01653v3.

[12] G. Cipolloni, L. Erdős, and D. Schröder, *Fluctuation around the circular law for random matrices with real entries,* preprint, https://arxiv.org/abs/2002.02438v1.

[13] M. Disertori, M. Lohmann, and S. Sodin, *The density of states of 1D random band matrices via a supersymmetric transfer operator,* preprint, https://arxiv.org/abs/1810.13150v1.

[14] M. Disertori, T. Spencer, and M. R. Zirnbauer, *Quasi-diffusion in a 3D supersymmetric hyperbolic sigma model,* Comm. Math. Phys. **300(2)** (2010), 435–486.

[15] K. Efetov, *Supersymmetry in Disorder and Chaos*, Cambridge University Press, Cambridge, 1997.

[16] K. B. Efetov, *Supersymmetry and theory of disordered metals,* Adv. in Physics **32(1)** (1983), 53–127.

[17] Y. V. Fyodorov and A. D. Mirlin, *Localization in ensemble of sparse random matrices,* Phys. Rev. Lett. **67** (1991), 2049–2052.

[18] Y. V. Fyodorov and E. Strahov, *An exact formula for general spectral correlation function of random Hermitian matrices. Random matrix theory*, J. Phys. A **36(12)** (2003), 3203–3214.

[19] J. Ginibre, *Statistical ensembles of complex, quaternion, and real matrices,* J. Math. Phys. **6** (1965), 440–449.

[20] V. L. Girko, *The circular law,* Teor. Veroyatn. Primen. **29(4)** (1984), 669–679.

[21] V. L. Girko, *The circular law: ten years later,* Random Oper. Stoch. Equ. **2(3)** (1994), 235–276.

[22] V. L. Girko, *The strong circular law. Twenty years later. I,* Random Oper. Stoch. Equ. **12(1)** (2004), 49–104.

[23] V. L. Girko, *The strong circular law. Twenty years later. II,* Random Oper. Stoch. Equ. **12(3)** (2004), 255–312.

[24] V. L. Girko, *The circular law. Twenty years later. III,* Random Oper. Stoch. Equ. **13(1)** (2005), 53–109.

[25] T. Guhr, *Supersymmetry,* The Oxford Handbook of Random Matrix Theory (eds. G. Akemann, J. Baik, and P.D. Francesco), Oxford university press, 2015, 135–154.

[26] P. Kopel, *Linear statistics of non-Hermitian matrices matching the real or complex Ginibre ensemble to four moments,* preprint, https://arxiv.org/abs/1510.02987v1

[27] P. Littelmann, H.-J. Sommers, and M.R. Zirnbauer, *Superbosonization of invariant random matrix ensembles,* Comm. Math. Phys., **283** (2008), 343–395.

[28] M.L. Mehta, *Random Matrices and the Statistical Theory of Energy Levels*, Academic Press, New York–London, 1967.

[29] A.D. Mirlin and Y.V. Fyodorov, *Universality of level correlation function of sparse random matrices,* J. Phys. A **24** (1991), 2273–2286.

[30] S. O'Rourke and D. Renfrew, *Central limit theorem for linear eigenvalue statistics of elliptic random matrices,* J. Theoret. Probab. **29** (2016), 1121–1191.

[31] L. Pastur and M. Shcherbina, *Szegö-Type theorems for one-dimensional Schrödinger operator with random potential (smooth case)* Zh. Mat. Fiz. Anal. Geom. **14** (2018), 362–388.

[32] C. Recher, M. Kieburg, T. Guhr, and M.R. Zirnbauer, *Supersymmetry approach to Wishart correlation matrices: Exact results,* J. Stat. Phys. **148(6)** (2012), 981–998.

[33] M. Shcherbina and T. Shcherbina, *Transfer matrix approach to 1d random band matrices: density of states,* J. Stat. Phys. **164(6)** (2016), 1233–1260.

[34] M. Shcherbina and T. Shcherbina, *Characteristic polynomials for 1D random band matrices from the localization side,* Comm. Math. Phys. **351(3)** (2017), 1009–1044.

[35] M. Shcherbina and T. Shcherbina, *Universality for 1d random band matrices: sigma-model approximation,* J. Stat. Phys. **172(2)** (2018), 627–664.

[36] T. Shcherbina, *On the correlation function of the characteristic polynomials of the Hermitian Wigner ensemble,* Comm. Math. Phys. **308** (2011), 1–21.

[37] T. Shcherbina, *On the correlation functions of the characteristic polynomials of the Hermitian sample covariance matrices,* Probab. Theory Related Fields **156** (2013), 449–482.

[38] E. Strahov and Y.V. Fyodorov, *Universal results for correlations of characteristic polynomials: Riemann-Hilbert approach,* Comm. Math. Phys. **241(2-3)** (2003), 343–382.

[39] T. Tao and V. Vu, *Random matrices: universality of ESDs and the circular law,* Ann. Probab. **38(5)** (2010), 2023–2065. With an appendix by Manjunath Krishnapur.

[40] T. Tao and V. Vu, *Random matrices: universality of local spectral statistics of non-Hermitian matrices,* Ann. Probab. **43(2)** (2015), 782–874.

[41] E.B. Vinberg, *A Course in Algebra*, American Mathematical Society, Providence, RI, 2003.

Received February 28, 2020.

Ievgenii Afanasiev,

B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, 47 Nauky Ave., Kharkiv, 61103, Ukraine,
E-mail: afanasiev@ilt.kharkov.ua

Про кореляційні функції характеристичних поліномів дійсних випадкових матриць з незалежними елементами

Ievgenii Afanasiev

У статті розглянуто кореляційні функції характеристичних поліномів дійсних випадкових матриць з незалежними елементами та встановлено асимптотичну поведінку цих кореляційних функцій у формі деякого інтеграла за інваріантною мірою по множині унітарних самодуальних матриць. Цей інтеграл обчислено для кореляційної функції другого порядку. З одержаної асимптотики випливає, що кореляційні функції ведуть себе таким же чином, як і у випадку дійсного ансамблю Жинібра з точністю до множника, що залежить лише від четвертого моменту спільного розподілу ймовірностей матричних елементів.

Ключові слова: теорія випадкових матриць, ансамбль Жинібра, кореляційні функції характеристичних поліномів, моменти характеристичних поліномів, суперсиметрія