Journal of Mathematical Physics, Analysis, Geometry
2020, Vol. 16, No. 3, pp. 283–290
doi: https://doi.org/10.15407/mag16.03.283

About Pogorelov's Method and Aleksandrov's Estimates

N.V. Krylov

We give a brief overview of the role of Aleksandrov's estimates and Pororelov's ideas in our research.

Key words: Aleksandrov's estimates, Pogorelov's method, Bellman's equations

Mathematical Subject Classification 2010: 35-01, 35-03, 51-01

During my fifth year at the Moscow State University E.B. Dynkin in his seminar was enthusiastically discussing a quite recent at that time theory of dynamic programming of controlled Markov chains. Imaging a discrete set and a random walk on it when at each time moment $0, 1, \ldots$ you can choose the probability distribution of the next step in a given in advance set of distributions. After the choice is made you are given a reward depending on you current position and the decision made and the chain goes to the next position according to the chosen distribution. The mathematical expectation of the cumulative reward for all times depend on the initial position and is called the value function. It turns out that the value function satisfies the so-called Bellman's equation, which is written in terms of finite differences.

The continuous analog of this setting leads to Bellman's equations which are fully nonlinear second-order possibly degenerate elliptic equations. Therefore, even in the continuous setting we had a perfect candidate for the solution: the value function of the corresponding controlled diffusion process. This was only a candidate, albeit a perfect one, because to prove that it is indeed a solution we needed to know at least that it is twice differentiable and even this turned out to be not a trivial task.

I decided to devote my years of post-graduate studies 1963–1966 to controlled diffusion processes and fully nonlinear equations and I did not succeed in achieving any progress. The point is that exercising an optimal control leads as a rule to Itô's stochastic equations with discontinuous coefficients and even the theory of stochastic equations with *continuous* coefficients was not developed at that time.

Nevertheless, I kept thinking about Bellman's equations. My enthusiasm became even greater when I asked O.A. Oleinik what is known about solvability of fully nonlinear equations and she answered "Nothing apart from the Monge–Ampère equations on the plane". Naturally, I asked myself if the Monge–Ampère equations are particular cases of Bellman's equations. And it turned out that they are!

By the way, in 1965 the book [2] by Bakelman just appeared.

A few years after I got my PhD degree I succeeded in proving that the value function for controlled diffusion processes in a very general case has two bounded derivatives. This, however, was still not enough to show that it solves indeed the corresponding Bellman's equation. Not only we needed Itô's formula for functions with two bounded, rather than continuous, derivatives but we also needed a kind of property showing that Green's functions of linear operators with no control on the smoothness of their coefficients so to speak cannot vanish on sets of positive measure.

Both properties would come almost for free if we knew that the Monge–Ampère equations in the multidimensional space have smooth solutions.

More details are as follows. Let $\mathbb{R}^d$ be a d-dimensional Euclidean space of points $x = (x^1, \dots, x^d)$, $B_r = \{x \in \mathbb{R}^d : |x| < r\}$, and let $\delta \in (0,1)$ be a fixed number. Let w_t, $t \geq 0$, be a d-dimensional Wiener process, and σ_t, $t \geq 0$, an appropriately measurable process with values in the set of $d \times d$-symmetric matrices whose eigenvalues are between δ and δ^{-1}. Define

$$x_t = \int_0^t \sigma_s \, dw_s \tag{1}$$

and let τ be the first exit time of x_t from B_1. What we needed is to know that there exists a constant N and $p \in (1, \infty)$ such that for any $f \geq 0$ and σ_t with the described properties

$$I := E \int_0^\tau f(x_t) \, dt \leq N \|f\|_{L_p(B_1)}. \tag{2}$$

Observe that if $\sigma_t = \sigma(x_t)$, so that (1), becomes a stochastic equation and $\sigma(x)$ is continuous, then $I = u(0)$, where u is a unique W^2_p-solution of

$$a^{ij} D_{ij} u = -f \tag{3}$$

in B_1 with zero boundary condition, where $a = (a^{ij}) = (1/2)\sigma^2$ and

$$D_{ij} = D_i D_j, \quad D_i = \frac{\partial}{\partial x^i}.$$

In that case (2) with N depending on the modulus of continuity of a and $p > d/2$ follows from the classical Sobolev space theory of elliptic equations. In the same case of a and $p \geq d$ estimate (2) with N depending only on d and δ is the famous Aleksandrov–Bakelman estimate (Aleksandrov 1960, Bakelman 1961). However for the needs of the theory of controlled diffusion processes we needed this estimate for processes σ_t depending on the past of x_t in an arbitrary way.

The Aleksandrov-Bakelman estimate is only good for the Markovian case: $\sigma_t = \sigma(x_t)$.

Since I was aware of the Monge–Ampère equations and works by Aleksandrov I knew an obvious way how to prove the estimate we needed if we knew that the solutions of the Monge–Ampère equation are smooth. The way was so obvious and short that it took me seven years to abandon the attempts to go around the missing smoothness properties and instead just repeat what Aleksandrov did at the level of convex polyhedra and discrete measures as right-hand sides of the simplest Monge–Ampère equation. This way was designed by Aleksandrov in [1] (1958) a few years before the Aleksandrov–Bakelman estimates appeared. That is why the estimates I proved for stochastic integrals I call Aleksandrov's estimates. They are not by any means the Aleksandrov–Bakelman or the Aleksandrov–Bakelman–Pucci estimates.

To give more details on the matter consider the following simplest Monge–Ampère equation

$$\det(-D_{ij}u) = f^d, \tag{4}$$

in B_1 with zero boundary condition on ∂B_1, where $f \geq 0$ is a given function. Equation (4) is considered only on concave functions u: $(D_{ij}u) \leq 0$. A simple argument based on the inequality between arithmetic and geometric means shows that (4) together with the condition $(D_{ij}u) \leq 0$ is equivalent to the single equation

$$\sup_{\substack{a\in A,\\ \operatorname{tr} a=1}} \left(\sum_{i,j=1}^{d} a^{ij} D_{ij}u + d\sqrt[d]{\det a}\, f \right) = 0, \tag{5}$$

where A is the set of symmetric nonnegative matrices $a = (a^{ij})$. Observe that (5) (and hence (4)) is a very particular case of Bellman's equations.

A.D. Aleksandrov in [1] obtained a very general result which implied the existence and uniqueness of concave solutions vanishing on ∂B_1 of (4) understood in a generalized sense. He also proved that for such a solution

$$u \leq N(d) \int_{B_1} f^d(x)\, dx, \tag{6}$$

with an explicit and sharp expression of $N(d)$. Now imagine that the solution u is smooth. Then for

$$a_t = (1/2)\sigma_t^2$$

by Itô's formula we have

$$u(0) = E \int_0^{\tau} \big(- \sum_{i,j=1}^{d} a_t^{ij} D_{ij}u(x_t)\big)\, dt.$$

Owing to (5) the integrand is greater than

$$d\sqrt[d]{\det a_t} f(x_t),$$

which along with (6) yields

$$dE\int_0^\tau \sqrt[d]{\det a_t}\, f(x_t)\,dt \le N(d)\int_{B_1} f^d(x)\,dx. \tag{7}$$

This proves (2) with $p = d$.

In 1971 I succeeded in publishing a self-contained proof of (7) (with not so sharp $N(d)$) by considering equation (4) in the space of convex polyhedra following the interpretation of Aleksandrov and a beautiful idea of Pogorelov which I learned about from the proof of Theorem 47 in Bakelman's book [2] and which, actually, is also explained in Aleksandrov's [1]. The same Pogorelov's idea was later used while deriving the Aleksandrov estimates in the parabolic case.

In a sense I was lucky having written this proof before 1971 because in 1971 A.V. Pogorelov published his results [6, 7] showing that the concave generalized solutions of (4) are indeed smooth. I was lucky not only because I got (7) by my methods and after Pogorelov's papers the above much shorter proof became available and I would not even think about writing my proof. There was another very disturbing reason. Imagine I used Pogorelov's results and built the whole rather vast theory of controlled diffusion processes based on them, as I did using my estimate in a book published in 1977, and then read in the same 1977 in the paper by Shiu Yuen Cheng and Shing Tung Yau [3] that the arguments of Pogorelov contain several very serious gaps. If somebody had discovered it while I was developing my theory taking as a base Pogorelov's results, my theory would go out of window and I would not get my Doctor of Sciences degree in 1973. Pogorelov's articles were published in Doklady, where the number of allowed pages were very restricted, and they, naturally, could not contain all details. So some criticism was unavoidable. In 1975 A.V. Pogorelov published his book [8] where he gave all details of the proofs. When Cheng and Yau criticized in 1977 his earlier work they were, probably, unaware of Pogorelov's book.

Anyhow, my way of proving Aleksandrov's estimate for stochastic integrals is way shorter and simpler than going first through [8] and then applying its results to deriving the estimate.

After estimate (7) was obtained and the theory of time homogeneous controlled diffusion processes was developed the next natural step was to consider time-inhomogeneous case. In that case a crucial role plays the estimate

$$E\int_0^\tau \sqrt[d+1]{\det a_t}\, f(t,x_t)\,dt \le N(d)\int_0^\infty\int_{B_1} f^{d+1}(t,x)\,dxdt. \tag{8}$$

Similarly to the above arguments about the simplest Monge–Ampère equation there was an idea to find a parabolic Monge–Ampère equation. In 1973 I suggested the following

$$\partial_t u \det(-D_{ij}u) = f^{d+1} \quad (\partial_t = \partial/\partial t) \tag{9}$$

in $C_1 := (0,\infty)\times B_1$ with zero boundary condition for $x \in \partial B_1$ and for $t = 0$. It was natural to consider (9) in the class of functions that are concave in x and increasing in t. In this class (9) is equivalent to the parabolic Bellman's equation

$$\sup_{(r,a)\in\Gamma}\Big(\sum_{i,j=1}^{d} a^{ij}D_{ij}u - r\partial_t u + (d+1)\sqrt[d+1]{r\det a}\,f\Big) = 0, \tag{10}$$

where $\Gamma = \{(r,a) : r \in (0,\infty), a \in A, r + \operatorname{tr} a = 1\}$. At that time there were no geometric interpretation of (9) and beautiful idea of Aleksandrov how to estimate the maximum of solution of the *elliptic* Monge–Ampère equation did not have any analogs.

However, the direct integration of both parts of (9) over C_1, using integration by parts and taking into account simple but very useful formulas

$$\sum_{i=1}^{d} D_iA_{ij} = \sum_{i=1}^{d} D_iA_{ji} = 0, \quad j = 1,\dots,d,$$

where A_{ij} are the co-factors of $D_{ij}u$ in $\det(-D_{ij}u)$, yields

$$\begin{aligned}
\int_C f^{d+1}\,d\,dt &= (1/d)\int_C \partial_t u\sum_{i,j=1}^{d} A_{ij}D_{ij}u\,dxdt\\
&= -(1/d)\int_C \sum_{i,j=1}^{d} A_{ij}(\partial_t D_i u)D_j u\,dx\,dt\\
&= (1/d)\int_C \sum_{i,j=1}^{d} A_{ij}(\partial_t D_{ij} u)u\,dx\,dt\\
&= (1/d)\int_C u\partial_t \det(-D_{ij}u)\,dx\,dt\\
&= (1/d)\int_{B_1} v\det(-D_{ij}v)\,dx - (1/d)\int_C f^{d+1}\,dx\,dt,
\end{aligned}$$

where $v(x) = u(\infty, x)$. Then a rather simple analytic fact is that the functional

$$\int_{B_1} v\det(-D_{ij}v)\,dx$$

is increasing on the set of concave v vanishing on ∂B_1 and this allows one to use the same cone as in Aleksandrov's elliptic estimate while estimating v and thus u.

Indeed, if $u \ge v$ are smooth concave functions vanishing on ∂B_1, then for $t \in [0,1]$ and

$$I_t = \int_{B_1} (tu + (1-t)v)\det\big(-D_{ij}(tu + (1-t)v)\big)\,dx$$

we have

$$\frac{d}{dt}I_t = \int_{B_1} (u-v)\det\big(-D_{ij}(tu + (1-t)v)\big)\,dx$$

$$+\int_{B_1}(tu+(1-t)v)D_{ij}(u-v)A_{ij}\,dx,$$

where the first integral is obviously positive and the second one equals d times the first one which is proved by integrating by parts. Hence $I_1 \geq I_0$, which is our analytic fact.

Again the trouble in this argument is that we did not know that (9) or (10) have sufficiently smooth solutions. Actually, after the theory of controlled diffusion processes was developed for elliptic case it occurred that it is possible to prove the basic estimate (8) using this theory and replacing (9) with a similar equation considered in all $\mathbb{R}^{d+1}$ on so-called λ-concave functions. Then we obtained a sufficiently smooth solution and performing the integration by parts similar to what is done above arrived at (8). Now the theory of controlled diffusion processes started to look somewhat entangled: first consider time-homogeneous processes, develop the theory guaranteeing that a modified equation (9) has a sufficiently smooth solution and thus obtain the basic estimate (8) and then by actually repeating what was done for time-homogeneous processes deal with the case of time-inhomogeneous ones.

I decided to avoid this circle by considering directly an analog of (9) on concave polyhedra in x piecewise linear and increasing in t. This lead to the following result.

Theorem 1. *Let Q be a convex, open, bounded set in $\mathbb{R}^d$, and let measures $\mu_0, \mu_1, \dots$ be concentrated in a closed set $F \subset Q$. Then there exists an increasing sequence of functions $z_0, z_1, \dots$ that are concave, continuous on $\bar{Q}$, equal to zero on ∂Q, and such that $z_0 = 0$ and in Q and for $i = 0, 1, \dots$ we have*

$$\mu_i(dx) = (z_{i+1}(x) - z_i(x))\omega(z_{i+1}, dx),$$

where, for concave z and set E, $\omega(z, E)$ is the volume of the normal image of E produced by z.

The proof of this result resides exclusively on the alluded before brilliant Pogorelov's idea of treating equations for polyhedra, when we first replace the μ_i's with measures concentrated on a finite number of points.

I submitted an article containing this result and its consequences for the theory of parabolic equations thus extending the Aleksandrov estimates in 1973 to Siberian Mathematical Journal. To me the result was a breakthrough, but I am not a geometer and it took three long years before it was published.

As a latest development in this story I state a recent result [5] published in 2019.

Theorem 2. *Let $a(x) = (a^{ij}(x))$ be a $d \times d$-symmetric nonnegative definite matrix-valued measurable function on B_1 such that* $\operatorname{tr} a > 0$ *in B_1. Let $\alpha \in [0, (d+1)/2)$ and $u \in W^2_{d,\mathrm{loc}}(B_1) \cap C(\bar{B}_1)$. Introduce*

$$Lu = a^{ij} D_{ij} u.$$

Then, for any $x_0 \in B$, $(0/0 := 0)$

$$u(x_0) \le \sup_{\partial B_1} u + N(d,\alpha)\psi^{\beta}(x_0)\Big(\int_{B_1} \psi^{\alpha} I_{Lu<0} (\det a)^{-1} |Lu|^d \, dx\Big)^{1/d}, \qquad (11)$$

where $\psi(x) = 1 - |x|^2$ *and* $\beta = (d + 1 - 2\alpha)/(2d)$.

Acknowledgments. The author would like to thank very much the organizers of the conference in honor of A.V. Pogorelov for inviting him.

References

[1] A.D. Aleksandrov, *Dirichlet's problem for the equation* $\det \|z_{ij}\| = \varphi(z_1, \ldots, z_n, z, x_1, \ldots, x_n)$*, I*, Vestnik Leningrad. Univ., Ser. Mat., Meh., Astr., **13** (1958), No. 1, 5–24 (Russian).

[2] I.Ja. Bakelman, *Geometric Methods of Solution of Elliptic Equations,* Nauka, Moscow, 1965 (Russian).

[3] Shiu Yuen Cheng and Shing Tung Yau, *On the regularity of the Monge–Ampère equation* $\det(\partial^2 u/\partial x_i \partial x_j) = F(x, u)$, Comm. Pure Appl. Math. **30** (1977), No. 1, 41–68.

[4] N.V. Krylov, *Sequences of convex functions and estimates of the maximum of the solution of a parabolic equation,* Sibirsk. Mat. Zh., **17** (1976), No. 2, 290–303 (Russian); Engl. transl.: Sib. Math. J., **17** (1976), No. 2, 226–236.

[5] N.V. Krylov, *Weighted Aleksandrov estimates: PDE and stochastic versions,* Algebra i Analiz **31** (2019), No. 3, 154–169.

[6] A.V. Pogorelov, *The regularity of the generalized solutions of the equation* $\det(\partial^2 u/\partial x^i \partial x^j) = \varphi(x^1, x^2, ..., x^n) > 0$, Dokl. Akad. Nauk SSSR **200** (1971), 534–537 (Russian); Engl. transl.: in Soviet Math. Dokl. **12** (1971), 1436–1440.

[7] A.V. Pogorelov, *The Dirichlet problem for the multidimensional analogue of the Monge–Ampère equation,* Dokl. Akad. Nauk SSSR **201** (1971), 790–793 (Russian); Engl. transl.: Soviet Math. Dokl. **12** (1971), 1227–1231.

[8] A.V. Pogorelov, *The Minkowski Multidimensional Problem,* Nauka, Moscow, 1975 (Russian); Engl. transl.: Scripta Series in Mathematics, V.H. Winston & Sons, Washington, D.C.; Halsted Press [John Wiley & Sons], New York-Toronto-London, 1978.

Received May 8, 2020.

N.V. Krylov,

127 Vincent Hall, University of Minnesota, Minneapolis, MN, 55455,

E-mail: nkrylov@umn.edu

Про метод Погорєлова і оцінки Александрова

N.V. Krylov

Ми даємо короткий огляд ролі оцінок Александрова та ідей Погорєлова у наших дослідженнях.

Ключові слова: оцінка Александрова, метод Погорєлова, рівняння Беллмана