

Natural Ricci Solitons on Tangent and Unit Tangent Bundles

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Considering pseudo-Riemannian g -natural metrics on tangent bundles, we prove that the condition of being Ricci soliton is hereditary in the sense that a Ricci soliton structure on the tangent bundle gives rise to a Ricci soliton structure on the base manifold. Restricting ourselves to some class of pseudo-Riemannian g -natural metrics, we show that the tangent bundle is a Ricci soliton if and only if the base manifold is flat and the potential vector field is a complete lift of a conformal vector field. We give then a classification of conformal vector fields on the tangent bundle of a flat Riemannian manifold equipped with these g -natural metrics. When unit tangent bundles over a constant curvature Riemannian manifold are endowed with pseudo-Riemannian Kaluza–Klein type metric, we give a classification of Ricci soliton structures whose potential vector fields are fiber-preserving, inferring the existence of some of them which are non Einstein.

Key words: tangent bundle, unit tangent (sphere) bundle, g -natural metrics, metrics of Kaluza–Klein type, Ricci solitons

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1. Introduction

Let M be a smooth manifold of dimension $n \geq 2$. A *Ricci soliton* on M is a triple (g, V, λ) , where g is a pseudo-Riemannian metric on M , Ric the associated Ricci tensor, V a vector field (called the *potential vector field*) and λ a real constant, satisfying the equation

$$\text{Ric} + \frac{1}{2}\mathcal{L}_V g = \lambda g, \quad (1.1)$$

\mathcal{L} being the Lie derivative. If there is a C^∞ -function on M such that $\text{Ric} + \nabla^2 f = \lambda g$ for some real constant λ , then (M, g) is said *gradient Ricci soliton*, where ∇ is the Levi-Civita connection of (M, g) . A Ricci soliton is said to be either *shrinking*, *steady*, or *expanding*, according on whether λ is negative, zero, or positive, respectively. Ricci solitons are a natural generalization of Einstein manifolds (an Einstein metric, together with a Killing vector field V , is a trivial Ricci soliton). Furthermore, they are the self-similar solutions to the Ricci flow.

Ricci solitons have been intensively studied in many contexts and from many points of view (we may refer to [16, 17] for details on the geometry of Ricci solitons). We are interested, in this paper, in the study of Ricci soliton structures in the framework of the geometry of tangent bundles. Indeed, Metrics on tangent and unit tangent sphere bundles have been an important source of examples in Differential Geometry. In particular, *g-natural metrics*, which generalize the *Sasaki metric* and still arise in a natural way from the metric of the base manifold, have been intensively studied during the last decades. Unlike the Sasaki metric which shows a very rigid behaviour, the large class of *g-natural metrics* provides examples for several different interesting geometric properties (cf. [3, 5, 7–9, 12, 14, 15, 21] and references therein).

In [2], the authors treated the problem of finding Ricci soliton structures on the unit tangent bundle of a Riemannian manifold, endowed with a pseudo-Riemannian *Kaluza–Klein type* metric, which is an interesting subclass of the class of *g-natural metrics* that shares with the Sasaki metric the property of preserving the orthogonality of horizontal and vertical distributions. They obtain a rigidity result in dimension three, showing that there are no nontrivial Ricci solitons among *g-natural metrics* of Kaluza–Klein type on the unit tangent sphere bundle of any Riemannian surface. On the other hand, they proved that, while Ricci solitons determined by tangential lifts remain trivial (i.e., an Einstein manifold) in arbitrary dimension, horizontal lifts of vector fields related to the geometry of flat base manifold (namely, homothetic vector fields) produce nontrivial Ricci solitons metrics of Kaluza–Klein type. They also gave a complete characterization of Gradient Ricci solitons of Kaluza–Klein type.

In this paper, we are interested in *natural Ricci soliton* structures on tangent and unit tangent bundles of Riemannian manifolds, i.e., those associated with pseudo-Riemannian *g-natural metrics*. Our purpose is twofold. On one hand, we investigate natural Ricci soliton structures on tangent bundles of Riemannian manifolds. We prove that every natural Ricci soliton structure on the tangent bundle gives rise to a Ricci soliton structure on the base manifold, confirming the “heridity” phenomenon of *g-natural metrics* (cf. [9]). Restricting ourselves *g-natural metrics* which are linear combinations of the three classical lifts (Sasaki, horizontal and vertical) of the base metric with constant coefficients, we give a complete characterization of Ricci soliton structures on the tangent bundle. We prove, in particular, that the existence of such structures requires the flatness of the base manifold, which constitutes a kind of rigidity of such metrics. Furthermore, this ensures the existence of non-trivial natural Ricci soliton structures on the tangent bundle of a Riemannian manifold.

On the other hand, we are looking for nontrivial natural Ricci solitons structures on the unit tangent bundle when the base manifold is of constant sectional curvature which is not necessarily zero. In this sense, we prove that the complete lift to the unit tangent bundle of a non-zero homothetic vector field on the base manifold is the potential vector field of a non trivial Ricci soliton structure on the unit tangent bundle endowed with an appropriately chosen pseudo-Riemannian Kaluza–Klein type metric. Furthermore, we shall give a complete classification of

fiber-preserving vector fields on the unit tangent bundle which are the potential vector fields of Ricci soliton structures on the unit tangent bundle endowed with a pseudo-Riemannian Kaluza–Klein type metric.

Finally, it is worth mentioning that the existence of natural Ricci soliton structures either on tangent bundles (Theorems 4.1 and 4.4) or on unit tangent bundles (Theorem 5.4) requires the existence of non-zero homothetic vector fields on the base manifold, which could presuppose some topological restrictions to the base manifold.

Hereafter, we will use the Einstein’ summation convention.

2. Preliminaries

2.1. Basic formulas on tangent bundles. Let (M, g) be an n -dimensional Riemannian manifold and ∇ the Levi-Civita connection of g . We shall denote by M_x the tangent space of M at a point $x \in M$. The tangent space of TM at any point $(x, u) \in TM$ splits into the horizontal and vertical subspaces with respect to ∇ :

$$(TM)_{(x,u)} = H_{(x,u)} \oplus V_{(x,u)}.$$

For $(x, u) \in TM$ and $X \in M_x$, there exists a unique vector $X^h \in H_{(x,u)}$ such that $p_*X^h = X$, where $p : TM \rightarrow M$ is the natural projection. We call X^h the *horizontal lift* of X to the point $(x, u) \in TM$. The *vertical lift* of a vector $X \in M_x$ to $(x, u) \in TM$ is the vector $X^v \in V_{(x,u)}$ such that $X^v(df) = Xf$ for all functions f on M . Here we consider 1-forms df on M as functions on TM (i.e., $(df)(x, u) = uf$).

Observe that the map $X \rightarrow X^h$ is an isomorphism between the vector spaces M_x and $H_{(x,u)}$. Similarly, the map $X \rightarrow X^v$ is an isomorphism between the vector spaces M_x and $V_{(x,u)}$. Obviously, each tangent vector $\tilde{Z} \in (TM)_{(x,u)}$ can be written in the form $\tilde{Z} = X^h + Y^v$, where $X, Y \in M_x$ are uniquely determined vectors.

Horizontal and vertical lifts of vector fields on M are defined in a corresponding way. Each system of local coordinates $\{(U; x^i, i = 1, \dots, n)\}$ in M induces on TM a system of local coordinates $\{(p^{-1}(U); x^i, u^i, i = 1, \dots, n)\}$. Let $X = \sum_i X^i \left(\frac{\partial}{\partial x^i}\right)_x$ be the local expression in $\{(U; x^i, i = 1, \dots, n)\}$ of a vector X in $M_x, x \in M$. Then, the horizontal lift X^h and the vertical lift X^v of X to $(x, u) \in TM$ are given, with respect to the induced coordinates, by:

$$X^h = \sum X^i \left(\frac{\partial}{\partial x^i}\right)_{(x,u)} - \sum \Gamma_{jk}^i u^j X^k \left(\frac{\partial}{\partial u^i}\right)_{(x,u)} \quad (2.1)$$

and

$$X^v = \sum X^i \left(\frac{\partial}{\partial u^i}\right)_{(x,u)}, \quad (2.2)$$

where $\left(\Gamma_{jk}^i\right)$ denote the Christoffel’s symbols of g .

Let $K : TTM \rightarrow TM$ be the connection map corresponding to the Levi-Civita connection ∇ of (M, g) . Note that K is characterized by $K(X^h) = 0$ and $K(X^v) = X$ for all $X \in TM$.

The *canonical vertical vector field* \mathcal{U} on TM is defined, in terms of local coordinates, by $\mathcal{U} = \sum_i u^i \partial / \partial u^i$. Here \mathcal{U} does not depend on the choice of local coordinates and is defined globally on TM . For a vector $u = \sum_i u^i (\partial / \partial x^i)_x \in M_x$, we see that $u_{(x,u)}^v = \sum_i u^i (\partial / \partial x^i)_{(x,u)}^v = \mathcal{U}_{(x,u)}$ and $u_{(x,u)}^h = \sum_i u^i (\partial / \partial x^i)_{(x,u)}^h$ (which is also known as the *geodesic vector field*).

There are three other interesting vector fields on the tangent bundle obtained by lifting operations of geometric objects on M : for any vector field X and a $(1, 1)$ -tensor field P on M , we define the vector fields X^c , ιP and $*P$ on TM , by

$$\begin{aligned} X_{(x,u)}^c &= X_{(x,u)}^h + (\nabla_u X)_{(x,u)}^v, \\ (\iota P)_{(x,u)} &= [P(u)]_{(x,u)}^v, \\ (*P)_{(x,u)} &= [P(u)]_{(x,u)}^h \end{aligned}$$

for all $(x, u) \in TM$. X^c is called the *complete lift* of X . It is easy to see that $X^c = X^h + \iota(\nabla X)$.

The Riemannian curvature R of g is defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}. \quad (2.3)$$

2.2. g -natural metrics on tangent bundles. There are three distinguished constructions of metrics on the tangent bundle TM

(a) the Sasaki metric g^s is defined by

$$\begin{aligned} g_{(x,u)}^s(X^h, Y^h) &= g(X, Y), & g_{(x,u)}^s(X^h, Y^v) &= 0, \\ g_{(x,u)}^s(X^v, Y^h) &= 0, & g_{(x,u)}^s(X^v, Y^v) &= g(X, Y) \end{aligned}$$

for all $X, Y \in M_x$;

(b) the horizontal lift g^h of g is a pseudo-Riemannian metric on TM , given by

$$\begin{aligned} g_{(x,u)}^h(X^h, Y^h) &= 0, & g_{(x,u)}^h(X^h, Y^v) &= g(X, Y), \\ g_{(x,u)}^h(X^v, Y^h) &= g(X, Y), & g_{(x,u)}^h(X^v, Y^v) &= 0 \end{aligned}$$

for all $X, Y \in M_x$;

(c) the vertical lift g^v of g is a degenerate metric on TM , given by

$$\begin{aligned} g_{(x,u)}^v(X^h, Y^h) &= g(X, Y), & g_{(x,u)}^v(X^h, Y^v) &= 0, \\ g_{(x,u)}^v(X^v, Y^h) &= 0, & g_{(x,u)}^v(X^v, Y^v) &= 0 \end{aligned}$$

for all $X, Y \in M_x$.

Starting from a Riemannian manifold (M, g) , a natural construction leads to introduce a wide class of metrics, called *g-natural*, on the tangent bundle TM [19], [20]. Such metrics are the image of g under first order natural operators $D : S_+^2 T^* \rightsquigarrow (S^2 T^*) T$, which transform Riemannian metrics on manifolds into metrics on their tangent bundles, where $S_+^2 T^*$ and $S^2 T^*$ denote the bundle functors of all Riemannian metrics and all symmetric $(0, 2)$ -tensors over n -manifolds respectively.

Given an arbitrary *g-natural* metric G on the tangent bundle TM of a Riemannian manifold (M, g) , there exist six smooth functions $\alpha_i, \beta_i : \mathbb{R}^+ \rightarrow \mathbb{R}$, $i = 1, 2, 3$, such that (see [9])

$$\left\{ \begin{array}{l} G_{(x,u)}(X^h, Y^h) = (\alpha_1 + \alpha_3)(r^2) g_x(X, Y) \\ \quad + (\beta_1 + \beta_3)(r^2) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^h, Y^v) = G_{(x,u)}(X^v, Y^h) \\ \quad = \alpha_2(r^2) g_x(X, Y) + \beta_2(r^2) g_x(X, u) g_x(Y, u), \\ G_{(x,u)}(X^v, Y^v) = \alpha_1(r^2) g_x(X, Y) + \beta_1(r^2) g_x(X, u) g_x(Y, u) \end{array} \right. \quad (2.4)$$

for every $u, X, Y \in M_x$, where $r^2 = g_x(u, u)$. Put

$$\begin{aligned} \phi_i(t) &= \alpha_i(t) + t\beta_i(t), \quad \alpha(t) = \alpha_1(t)(\alpha_1 + \alpha_3)(t) - \alpha_2^2(t), \\ \phi(t) &= \phi_1(t)(\phi_1 + \phi_3)(t) - \phi_2^2(t) \end{aligned}$$

for all $t \in \mathbb{R}^+$. It is easily seen that G is

- non-degenerate if and only if

$$\alpha(t) \neq 0, \quad \phi(t) \neq 0 \quad \text{for all } t \in \mathbb{R}^+;$$

- Riemannian if and only if

$$\alpha_1(t) > 0, \quad \phi_1(t) > 0, \quad \alpha(t) > 0, \quad \phi(t) > 0 \quad \text{for all } t \in \mathbb{R}^+.$$

The wide class of *g-natural* metrics includes several well known metrics (Riemannian and not) on TM . In particular:

- the *Sasaki metric* g_S is obtained for $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0$;
- *Kaluza–Klein metrics*, as commonly defined on principal bundles (see, e. g., [23]), are obtained for $\alpha_2 = \beta_2 = \beta_1 + \beta_3 = 0$;
- *metrics of Kaluza–Klein type* are defined by the geometric condition of orthogonality between horizontal and vertical distributions [13, 15]. Thus, a *g-natural* metric G is of Kaluza–Klein type if $\alpha_2 = \beta_2 = 0$.

2.3. g -natural metrics on unit tangent bundles. The *unit tangent bundle* over a Riemannian manifold (M, g) is the hypersurface of TM , given by

$$T_1M = \{(x, u) \in TM \mid g_x(u, u) = 1\}.$$

We will denote by $p_1 : T_1M \rightarrow M$ the bundle projection. The tangent space of T_1M at a point $(x, u) \in T_1M$ is given by

$$(T_1M)_{(x,u)} = \{X^h + Y^v/X \in M_x, Y \in \{u\}^\perp \subset M_x\}. \quad (2.5)$$

By definition, *g -natural metrics on the unit tangent bundle* are the metrics induced on the hypersurface T_1M by corresponding g -natural metrics on TM . As proved in [7] for the Riemannian case, and extended to pseudo-Riemannian settings in [12], they are completely determined by the values of the four real constants

$$a := \alpha_1(1), \quad b := \alpha_2(1), \quad c := \alpha_3(1), \quad d := (\beta_1 + \beta_3)(1)$$

and admit the following explicit description.

Theorem 2.1 ([7]). *Let (M, g) be a Riemannian manifold. For every pseudo-Riemannian metric \tilde{G} on T_1M induced from a g -natural metric G on TM , there exist four constants a, b, c and d , satisfying the inequalities*

$$a \neq 0, \quad \alpha := a(a+c) - b^2 \neq 0, \quad \varphi := a+c+d \neq 0$$

(in particular, they are Riemannian if $a, \alpha, \varphi > 0$), such that

$$\begin{cases} \tilde{G}_{(x,u)}(X_1^h, X_2^h) = (a+c)g_x(X_1, X_2) + dg_x(X_1, u)g_x(X_2, u), \\ \tilde{G}_{(x,u)}(X_1^h, Y_1^v) = bg_x(X_1, Y_1), \\ \tilde{G}_{(x,u)}(Y_1^v, Y_2^v) = ag_x(Y_1, Y_2) \end{cases} \quad (2.6)$$

for all $(x, u) \in T_1M$, $X_1, X_2 \in M_x$ and $Y_1, Y_2 \in \{u\}^\perp \subset M_x$.

In particular, the *Sasaki metric* on T_1M corresponds to the case where $a = 1$ and $b = c = d = 0$; *Kaluza–Klein metrics* are obtained when $b = d = 0$; *metrics of Kaluza–Klein type* are given by the case $b = 0$. In the remaining part of the paper, we shall restrict ourselves to the latter type of g -natural metrics on T_1M .

Since G must be nondegenerate, throughout the paper we implicitly assume that inequalities $a \neq 0$, $a+c \neq 0$ and $\varphi := a+c+d \neq 0$ hold (they are exactly the special case of inequalities from Theorem 2.1 for $b = 0$).

For any vector field Z on TM , we define its tangential component $t\{Z\}$ at points of T_1M , by $t\{Z\} := Z|_{T_1M} - G(Z|_{T_1M}, N)N$, obtaining a vector field on T_1M . When Z is the horizontal, the vertical or the complete lift of a vector field X on M or the vector field ιP for a $(1,1)$ -tensor field P on M , then we obtain, respectively, the following special vector fields on T_1M :

- a) the *horizontal lift* to T_1M of X , denoted also X^h , and given by $X^h := X^h|_{T_1M}$, since X^h is orthogonal to N everywhere on T_1M ;
- b) the *tangential lift* X^t with respect to G of X , defined as the tangential component vector field of the vertical lift X^v of X , that is,

$$X_{(x,u)}^t = [X - g_x(u, X)u]_{(x,u)}^v + \frac{b}{\varphi} g_x(u, X) u_{(x,u)}^h \quad (2.7)$$

for all $(x, u) \in T_1M$. If $X_x \in M_x$ is orthogonal to u , then $X_{(x,u)}^t = X_{(x,u)}^v$;

- c) the *complete lift* to T_1M of X , denoted by $X^{\bar{c}}$, is given by

$$X_{(x,u)}^{\bar{c}} = X_{(x,u)}^h + (\nabla_u X)_{(x,u)}^t \quad (2.8)$$

for all $(x, u) \in T_1M$. It is worth mentioning that the complete lift X^c on TM , restricted to T_1M , coincides with $X^{\bar{c}}$ if and only if X is a Killing vector field on M ;

- d) the vector field $\tilde{t}P$ on T_1M , given by

$$(\tilde{t}P)_{(x,u)} = [P(u)]_{(x,u)}^t \quad (2.9)$$

for all $(x, u) \in T_1M$.

We will make use of the following relations, which are easy to check:

$$X^h(f \circ p_1) = X(f) \circ p_1, \quad (2.10)$$

$$X^t(f \circ p_1) = 0, \quad (2.11)$$

$$X_{(x,u)}^h(g(Y, \cdot)) = g(\nabla_{X_x} Y, u) \quad (2.12)$$

$$X_{(x,u)}^t(g(Y, \cdot)) = g(X_x, Y_x) - g(X_x, u)g(Y_x, u) \quad (2.13)$$

for all $X, Y \in \mathfrak{X}(M)$, $f \in C^\infty(M)$ and $(x, u) \in T_1M$.

Convention 2.2. Because $u_{(x,u)}^t = 0$ for $(x, u) \in T_1M$, which means that the tangent space $(T_1M)_{(x,u)}$ coincides with the set

$$\left\{ X^h + Y^t/X \in M_x, Y \in \{u\}^\perp \subset M_x \right\}, \quad (2.14)$$

the operation of tangential lift from M_x to a point $(x, u) \in T_1M$ will be always applied only to vectors of M_x which are orthogonal to u .

Assuming that (M, g) is of constant curvature κ for a g -natural metric \tilde{G} of Kaluza–Klein type, the following formulas are deduced from the expressions of the Levi-Civita connection of (T_1M, \tilde{G}) given in [6]:

Proposition 2.3. *If (M, g) is a Riemannian manifold of constant sectional curvature κ and \tilde{G} is a pseudo-Riemannian g -natural metric of Kaluza–Klein type on T_1M , then the Levi-Civita connection $\tilde{\nabla}$ of (T_1M, \tilde{G}) is given by*

$$\begin{aligned} \left(\tilde{\nabla}_{X^h} Y^h\right)_{(x,u)} &= \{\nabla_X Y\}_{(x,u)}^h + \frac{1}{2a} \left\{ - (a\kappa + d) g(Y_x, u) X_x \right. \\ &\quad \left. + (a\kappa - d) g(X_x, u) Y_x \right\}_{(x,u)}^t, \\ \left(\tilde{\nabla}_{X^h} Y^t\right)_{(x,u)} &= \left\{ \frac{d - a\kappa}{2(a+c)} g(X_x, u) Y_x + \frac{1}{2\varphi} \left[(a\kappa + d) g(X_x, Y_x) \right. \right. \\ &\quad \left. \left. + \frac{d(a\kappa + d - 2\varphi)}{a+c} g(X_x, u) g(Y_x, u) \right] u \right\}_{(x,u)}^h + \{\nabla_X Y\}_{(x,u)}^t, \\ \left(\tilde{\nabla}_{X^t} Y^h\right)_{(x,u)} &= \left\{ \frac{d - a\kappa}{2(a+c)} g(Y_x, u) X_x + \frac{1}{2\varphi} \left[(a\kappa + d) g(X_x, Y_x) \right. \right. \\ &\quad \left. \left. + \frac{d(a\kappa + d - 2\varphi)}{a+c} g(X_x, u) g(Y_x, u) \right] u \right\}_{(x,u)}^h, \\ \left(\tilde{\nabla}_{X^t} Y^t\right)_{(x,u)} &= - \{g(Y_x, u) X_x\}_{(x,u)}^t \end{aligned}$$

for all $x \in M$, $(x, u) \in T_1M$ and all arbitrary vector fields $X, Y \in \mathfrak{X}(M)$.

Using Proposition 2.3, we can prove the following lemma.

Lemma 2.4. *For all $\xi \in \mathfrak{X}(M)$, $(x, u) \in TM$ and $X, Y \in T_xM$, we have*

$$\begin{aligned} \left(\mathcal{L}_{\xi^t} \tilde{G}\right)_{(x,u)} \left(X^h, Y^h\right) &= d\{g(\xi_x, X) g(Y, u) + g(\xi_x, Y) g(X, u) \\ &\quad - 2g(\xi_x, u) g(X, u) g(Y, u)\}, \\ \left(\mathcal{L}_{\xi^t} \tilde{G}\right)_{(x,u)} \left(X^h, Y^t\right) &= a\{g(\nabla_X \xi, Y) - g(Y, u) g(\nabla_X \xi, u)\}, \\ \left(\mathcal{L}_{\xi^t} \tilde{G}\right)_{(x,u)} \left(X^t, Y^t\right) &= -2ag(\xi_x, u) \{g(X, Y) - g(X, u) g(Y, u)\}. \end{aligned}$$

Lemma 2.5. *For all $\xi \in \mathfrak{X}(M)$, $(x, u) \in TM$ and $X, Y \in T_xM$, we have*

$$\begin{aligned} \left(\mathcal{L}_{\xi^h} \tilde{G}\right)_{(x,u)} \left(X^h, Y^h\right) &= (a+c) \{g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)\}, \\ &\quad + d\{g(\nabla_X \xi, u) g(Y, u) + g(\nabla_Y \xi, u) g(X, u)\}, \\ \left(\mathcal{L}_{\xi^h} \tilde{G}\right)_{(x,u)} \left(X^h, Y^t\right) &= a\kappa\{g(\xi_x, Y) g(X, u) - g(\xi_x, u) g(X, Y)\}, \\ \left(\mathcal{L}_{\xi^h} \tilde{G}\right)_{(x,u)} \left(X^t, Y^t\right) &= 0. \end{aligned}$$

Next, the Ricci tensor $\widetilde{\text{Ric}}$ of (T_1M, \tilde{G}) is given by the following.

Proposition 2.6 ([8]). *If (M, g) is a space of constant sectional curvature κ and \tilde{G} is a pseudo-Riemannian g -natural metric of Kaluza–Klein type on T_1M , then the Ricci tensor $\widetilde{\text{Ric}}$ of (T_1M, \tilde{G}) is given by*

$$\begin{aligned} \widetilde{\text{Ric}}_{(x,u)}(X^h, Y^h) &= \frac{1}{2a\varphi} [-a^2\kappa^2 + 2(n-1)a\varphi\kappa + d(d-2\varphi)] g(X, Y) \\ &\quad + \frac{1}{2a(a+c)\varphi} [-a^2((n-2)\varphi + d)\kappa^2 + d(2n(a+c)\varphi \\ &\quad + (n-1)d\varphi - d(a+c))] g(X, u) g(Y, u), \end{aligned} \quad (2.15)$$

$$\widetilde{\text{Ric}}_{(x,u)}(X^h, Y^t) = 0, \quad (2.16)$$

$$\widetilde{\text{Ric}}_{(x,u)}(X^t, Y^t) = \frac{1}{2(a+c)\varphi} [a^2\kappa^2 + 2(n-2)(a+c)\varphi - d^2] g(X, Y) \quad (2.17)$$

for all $x \in M$, $(x, u) \in T_1M$ and all arbitrary vectors $X, Y \in M_x$ satisfying Convention 2.2.

3. Natural Ricci solitons on tangent bundles

Theorem 3.1. *Let (M, g) be a Riemannian manifold of dimension $n \geq 3$ and G be a pseudo-Riemannian g -natural metric on TM whose functions α_i, β_i $i = 1, 2, 3$ satisfy $\alpha_1(0)(\alpha_1 + \alpha_3)(0) - 2\alpha_2^2(0) \neq 0$. If $(TM, G, Z, \bar{\lambda})$ is a Ricci Soliton, then (M, g, Z_0, λ) is a Ricci soliton with*

$$\begin{aligned} Z_0(x) &= \frac{\alpha(0)}{[\alpha_1(\alpha_1 + \alpha_3) - 2\alpha_2^2](0)} \{(\alpha_1 + \alpha_3)(0)d\pi(Z(x, 0)) + \alpha_2(0)K(Z(x, 0))\}; \\ \lambda &= \frac{(\alpha_1 + \alpha_3)[\bar{\lambda}\alpha + \beta_1 + \beta_3 + n(\alpha_1 + \alpha_3)']}{\alpha_1(\alpha_1 + \alpha_3) - 2\alpha_2^2}(0), \end{aligned}$$

where K is the connection map.

Proof. Fix $x \in M$ and let (E_1, \dots, E_n) be an orthonormal basis of (M_x, g_x) . We will distinguish two cases:

Case 1: $(\alpha_1 + \alpha_3)(0) \neq 0$. If we put

$$\begin{aligned} F_i &= \frac{1}{\sqrt{|\alpha_1 + \alpha_3(0)|}} E_i^h(x, 0), \\ F_{i+n} &= \frac{1}{\sqrt{|\alpha(0)(\alpha_1 + \alpha_3)(0)|}} \left[\alpha_2(0) E_i^h(x, 0) - (\alpha_1 + \alpha_3)(0) E_i^v(x, 0) \right], \end{aligned}$$

$i = 1, \dots, n$, then we obtain an orthonormal basis $\{F_1, \dots, F_{2n}\}$ of the tangent space $((TM)_{(x,0)}, G_{(x,0)})$. If we put $\varepsilon_I := G(F_I, F_I)$, $I = 1, \dots, 2n$, then the Ricci curvature of (TM, G) at $(x, 0)$ is then given by

$$\overline{\text{Ric}}_{(x,0)}(V, W) = \sum_{I=1}^{2n} \varepsilon_I G_{(x,0)}(\bar{R}(V, F_I) F_I, W)$$

for every $V, W \in (TM)_{(x,0)}$.

Taking $V = X^h$ and $W = Y^h$ for $X, Y \in M_x$, then simple calculation gives

$$\begin{aligned} \overline{\text{Ric}}_{(x,0)}(X^h, Y^h) &= \frac{1}{\alpha(0)} \sum_{i=1}^n \left\{ \alpha_1(0) G_{(x,0)} \left(\bar{R}(X^h, E_i^h) E_i^h, Y^h \right) \right. \\ &\quad + (\alpha_1 + \alpha_3)(0) G_{(x,0)} \left(\bar{R}(X^h, E_i^v) E_i^v, Y^h \right) \\ &\quad - \alpha_2(0) \left[G_{(x,0)} \left(\bar{R}(X^h, E_i^h) E_i^v, Y^h \right) \right. \\ &\quad \left. \left. + G_{(x,0)} \left(\bar{R}(X^h, E_i^v) E_i^h, Y^h \right) \right] \right\}. \end{aligned} \quad (3.1)$$

But, restricting the Riemannian curvature \bar{R} of (TM, G) to the zero section (cf. [9, Proposition 3.1]), we obtain

$$\begin{aligned} G_{(x,0)} \left(\bar{R}(X^h, E_i^h) E_i^h, Y^h \right) &= (\alpha_1 + \alpha_3)(0) g(R(X, E_i) E_i, Y), \\ G_{(x,0)} \left(\bar{R}(X^h, E_i^h) E_i^v, Y^h \right) &= \alpha_2(0) g(R(X, E_i) E_i, Y), \\ G_{(x,0)} \left(\bar{R}(X^h, E_i^v) E_i^h, Y^h \right) &= \alpha_2(0) g(R(X, E_i) E_i, Y), \\ G_{(x,0)} \left(\bar{R}(X^h, E_i^v) E_i^v, Y^h \right) &= -(\alpha_1 + \alpha_3)'(0) g(X, Y) \\ &\quad - (\beta_1 + \beta_3)(0) g(X, E_i) g(Y, E_i). \end{aligned}$$

Replacing from the last identities into (3.1), we deduce that

$$\begin{aligned} \overline{\text{Ric}}_{(x,0)}(X^h, Y^h) &= \frac{1}{\alpha(0)} \sum_{i=1}^n \left\{ [\alpha_1(\alpha_1 + \alpha_3) - 2\alpha_2^2](0) g(R(X, E_i) E_i, Y) \right. \\ &\quad - (\alpha_1 + \alpha_3)(0) [(\alpha_1 + \alpha_3)'(0) g(X, Y) \\ &\quad \left. + (\beta_1 + \beta_3)(0) g(X, E_i) g(Y, E_i)] \right\}. \end{aligned}$$

Since (E_1, \dots, E_n) is an orthonormal basis of (M_x, g_x) , we deduce that

$$\begin{aligned} \overline{\text{Ric}}_{(x,0)}(X^h, Y^h) &= \frac{1}{\alpha(0)} [((\alpha_1 + \alpha_3) - 2\alpha_2^2)(0) \text{Ric}(X, Y) \\ &\quad - (\alpha_1 + \alpha_3)(0) ((\beta_1 + \beta_3)(0) \\ &\quad + n(\alpha_1 + \alpha_3)'(0)) g(X, Y)]. \end{aligned} \quad (3.2)$$

$$+ n(\alpha_1 + \alpha_3)'(0)) g(X, Y)]. \quad (3.3)$$

Now, if $(TM, G, Z, \bar{\lambda})$ is a Ricci Soliton, then we have in particular:

$$\bar{\text{Ric}}_{(x,0)}(X^h, Y^h) + \frac{1}{2} (\mathcal{L}_Z G)_{(x,0)}(X^h, Y^h) = \bar{\lambda} G_{(x,0)}(X^h, Y^h). \quad (3.4)$$

Expressing Z , in an induced local system $(p^{-1}(U); x^i, u^i, i = 1, \dots, n)$ of TM , as $Z = \sum_{l=1}^n [A^l \frac{\partial}{\partial x^l} + B^l \frac{\partial}{\partial u^l}]$, and restricting the Levi-Civita connection $\bar{\nabla}$ of (TM, G) to the zero section (cf. [9, Proposition 1.5]), we obtain

$$(\mathcal{L}_Z G)_{(x,0)}(X^h, Y^h) = G_{(x,0)}(X^h, \bar{\nabla}_{Y^h} Z) + G_{(x,0)}(Y^h, \bar{\nabla}_{X^h} Z)$$

$$\begin{aligned}
&= (\alpha_1 + \alpha_3)(0) \left[g \left(X, Y \left(A^l(x, 0) \right) \frac{\partial}{\partial x^l} \Big|_x \right) \right. \\
&\quad + g \left(Y, X \left(A^l(x, 0) \right) \frac{\partial}{\partial x^l} \Big|_x \right) \\
&\quad \left. + A^l(x, 0) \left(g \left(X, \nabla_Y \frac{\partial}{\partial x^l} \right) + g \left(Y, \nabla_X \frac{\partial}{\partial x^l} \right) \right) \right] \\
&\quad + \alpha_2(0) \left[g \left(X, Y \left(B^l(x, 0) \right) \frac{\partial}{\partial x^l} \Big|_x \right) \right. \\
&\quad + g \left(Y, X \left(B^l(x, 0) \right) \frac{\partial}{\partial x^l} \Big|_x \right) \\
&\quad \left. + B^l(x, 0) \left(g \left(X, \nabla_Y \frac{\partial}{\partial x^l} \right) + g \left(Y, \nabla_X \frac{\partial}{\partial x^l} \right) \right) \right]. \quad (3.5)
\end{aligned}$$

Setting, for every $x \in U$,

$$W(x) = \left[(\alpha_1 + \alpha_3)(0) A^l(x, 0) + \alpha_2(0) B^l(x, 0) \right] \frac{\partial}{\partial x^l} \Big|_x,$$

then W is a vector field on U , and (3.5) becomes

$$(\mathcal{L}_Z G)_{(x,0)}(X^h, Y^h) = (\mathcal{L}_W g)_x(X, Y). \quad (3.6)$$

Substituting from (3.2) and (3.6) into (3.4), we deduce, under

$$(\alpha_1(\alpha_1 + \alpha_3) - 2\alpha_2^2)(0) \neq 0,$$

that

$$\text{Ric}(X, Y) + \frac{1}{2} \mathcal{L}_{Z_0} g(X, Y) = \lambda g(X, Y)$$

for every $X, Y \in M_x$, where

$$Z_0 = \frac{\alpha(0)}{(\alpha_1(\alpha_1 + \alpha_3) - 2\alpha_2^2)(0)} W$$

and

$$\lambda = \frac{(\alpha_1 + \alpha_3) [\bar{\lambda}\alpha + \beta_1 + \beta_3 + n(\alpha_1 + \alpha_3)']}{\alpha_1(\alpha_1 + \alpha_3) - 2\alpha_2^2}(0).$$

Case 2: $(\alpha_1 + \alpha_3)(0) = 0$. In this case $\alpha_2(0) \neq 0$. We put

$$F_i = \frac{1}{\sqrt{2|\alpha_2(0)|}} [E_i^h + E_i^v], \quad \text{and} \quad F_{n+i} = \frac{1}{\sqrt{2|\alpha_2(0)|}} [E_i^h - E_i^v],$$

$i = 1, \dots, n$. As in the first case, $\{F_I; I = 1, \dots, 2n\}$ is an orthonormal basis of $((TM)_{(x,0)}, G_{(x,0)})$, and in a similar way we obtain

$$\overline{\text{Ric}}_{(x,0)}(X^h, Y^h) = 2\text{Ric}(X, Y),$$

and then $\text{Ric}(X, Y) + \frac{1}{2}\mathcal{L}_{Z'_0}g(X, Y) = 0$ for any $x \in M$ and $X, Y \in M_x$, where $Z'_0(x) := -\alpha_2(0)B^l(x, 0)\frac{\partial}{\partial x^l}\Big|_x$, i.e. $(TM, G, Z'_0, 0)$ is a Ricci Soliton. Remark that, for $(\alpha_1 + \alpha_3)(0) = 0$, we obtain $\bar{\lambda} = 0$ and $Z_0 = Z'_0$. This completes our proof. \square

Corollary 3.2. *Let (M, g) be a Riemannian manifold of dimension $n \geq 3$ and G be a pseudo-Riemannian g -natural metric on TM whose functions α_i, β_i $i = 1, 2, 3$ satisfy $\alpha_1(0)(\alpha_1 + \alpha_3)(0) - 2\alpha_2^2(0) \neq 0$. If $(TM, G, \bar{f}, \bar{\lambda})$ is a gradient type Ricci soliton, then (M, g, f, λ) is a gradient type Ricci soliton with*

$$f(x) = \frac{\alpha(0)}{\alpha_1(0)(\alpha_1 + \alpha_3)(0) - 2\alpha_2^2(0)}\bar{f}(x, 0) \quad \text{for all } x \in M;$$

$$\lambda = \frac{(\alpha_1 + \alpha_3)[\bar{\lambda}\alpha + \beta_1 + \beta_3 + n(\alpha_1 + \alpha_3)']}{\alpha_1(\alpha_1 + \alpha_3) - 2\alpha_2^2}(0).$$

4. g -natural metrics of the form $ag^s + bg^h + cg^v$

Theorem 4.1. *Let (M, g) be a Riemannian manifold of dimension $n \geq 2$, $G = ag^s + bg^h + cg^v$ such that $a \neq 0$, $a + c \neq 0$ and $a(a + c) - b^2 \neq 0$, Z be a vector field on TM and $\bar{\lambda} \in \mathbb{R}$. Then $(TM, G, Z, \bar{\lambda})$ is a Ricci Soliton if and only if*

1. (M, g) flat;
2. Z is a homothetic vector field on (TM, G) , with homothety factor $\bar{\lambda}$.

Proof. Let $(TM, G, Z, \bar{\lambda})$ be a Ricci Soliton. For x in M , let (x^1, \dots, x^n) be a normal coordinate system at x , and $(\pi^{-1}(U), x^1, \dots, x^n, u^1, \dots, u^n)$ be its induced coordinate system on TM . With respect to these induced coordinates, Z can be expressed as $Z = \sum_{l=1}^n [A^l \frac{\partial}{\partial x^l} + B^l \frac{\partial}{\partial u^l}]$. We have in particular

$$\overline{\text{Ric}}\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) + \frac{1}{2}(\mathcal{L}_Z G)\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) = \bar{\lambda}G\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) \quad (4.1)$$

for $i, j = 1, \dots, n$. Using the expression of the Levi-Civita connection (cf. [10, p.32]) and the fact that (x^1, \dots, x^n) is normal at x , we have

$$\begin{aligned} (\mathcal{L}_Z G)_{(x,u)}\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) &= G_{(x,u)}\left(\frac{\partial}{\partial u^i}, \bar{\nabla}_{\frac{\partial}{\partial u^j}} Z\right) + G_{(x,u)}\left(\frac{\partial}{\partial u^j}, \bar{\nabla}_{\frac{\partial}{\partial u^i}} Z\right) \\ &= \frac{\partial}{\partial u^j}\Big|_{(x,u)}(bA^i + aB^i) + \sum_{l=1}^n A^l G_{(x,u)}\left(\frac{\partial}{\partial u^i}, \bar{\nabla}_{\frac{\partial}{\partial u^j}}\left(\frac{\partial}{\partial x^l}\right)^h\right) \\ &+ \frac{\partial}{\partial u^i}\Big|_{(x,u)}(bA^j + aB^j) + \sum_{l=1}^n A^l G_{(x,u)}\left(\frac{\partial}{\partial u^j}, \bar{\nabla}_{\frac{\partial}{\partial u^i}}\left(\frac{\partial}{\partial x^l}\right)^h\right) \\ &= \frac{\partial}{\partial u^j}\Big|_{(x,u)}(bA^i + aB^i) + \frac{\partial}{\partial u^i}\Big|_{(x,u)}(bA^j + aB^j) \end{aligned}$$

for all $(x, u) \in M_x$. On the other hand, using [10, Proposition 3.1] and the fact that (x^1, \dots, x^n) is normal at x , we have

$$\overline{\text{Ric}}_{(x,u)} \left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right) = \frac{a^4}{4\alpha^2} \sum_{l=1}^n g \left(R \left(u, \frac{\partial}{\partial x^i} \Big|_x \right) \frac{\partial}{\partial x^l} \Big|_x, R \left(u, \frac{\partial}{\partial x^j} \Big|_x \right) \frac{\partial}{\partial x^l} \Big|_x \right).$$

So we get

$$\begin{aligned} & \frac{\partial}{\partial u^j} \Big|_{(x,u)} (bA^i + aB^i) + \frac{\partial}{\partial u^i} \Big|_{(x,u)} (bA^j + aB^j) \\ &= 2 \left\{ \bar{\lambda} a \delta_{ij} - \frac{a^4}{4\alpha^2} \sum_{l=1}^n g \left(R \left(u, \frac{\partial}{\partial x^i} \Big|_x \right) \frac{\partial}{\partial x^l} \Big|_x, R \left(u, \frac{\partial}{\partial x^j} \Big|_x \right) \frac{\partial}{\partial x^l} \Big|_x \right) \right\}. \end{aligned} \quad (4.2)$$

Let us define the functions D_i , $i = 1, \dots, n$, on M_x by

$$D_i = \frac{2\alpha^2}{a^4} [\bar{\lambda} a u^i - (bA^i + aB^i)].$$

So (4.2) reduces to

$$\frac{\partial D_i}{\partial u^j} \Big|_{(x,u)} + \frac{\partial D_j}{\partial u^i} \Big|_{(x,u)} = \sum_{l=1}^n g \left(R \left(u, \frac{\partial}{\partial x^i} \Big|_x \right) \frac{\partial}{\partial x^l} \Big|_x, R \left(u, \frac{\partial}{\partial x^j} \Big|_x \right) \frac{\partial}{\partial x^l} \Big|_x \right), \quad (4.3)$$

where $i, j = 1, \dots, n$.

For $i = j$, we get

$$\begin{aligned} \frac{\partial D_i}{\partial u^i} \Big|_{(x,u)} &= \sum_{l=1}^n \frac{1}{2} g \left(R \left(u, \frac{\partial}{\partial x^i} \Big|_x \right) \frac{\partial}{\partial x^l} \Big|_x, R \left(u, \frac{\partial}{\partial x^i} \Big|_x \right) \frac{\partial}{\partial x^l} \Big|_x \right) \\ &= \frac{1}{2} \sum_{l=1}^n \sum_{\substack{s \neq i, t \neq i}} u^s u^t g \left(R \left(\frac{\partial}{\partial x^s} \Big|_x, \frac{\partial}{\partial x^i} \Big|_x \right) \frac{\partial}{\partial x^l} \Big|_x, R \left(\frac{\partial}{\partial x^t} \Big|_x, \frac{\partial}{\partial x^i} \Big|_x \right) \frac{\partial}{\partial x^l} \Big|_x \right). \end{aligned}$$

Hence

$$\begin{aligned} D_i &= \frac{1}{2} \sum_{l=1}^n \sum_{\substack{s \neq i \\ t \neq i, r}} u^s u^t u^r \\ &\quad \times g \left(R \left(\frac{\partial}{\partial x^s} \Big|_x, \frac{\partial}{\partial x^r} \Big|_x \right) \frac{\partial}{\partial x^l} \Big|_x, R \left(\frac{\partial}{\partial x^t} \Big|_x, \frac{\partial}{\partial x^r} \Big|_x \right) \frac{\partial}{\partial x^l} \Big|_x \right) + f_i, \end{aligned}$$

with f_i is a C^∞ function on M_x which does not depend on u^i .

For $i \neq j$, deriving equation (4.3) twice with respect to u_j and u_i and summing up, we obtain

$$\frac{\partial^2}{\partial^2 u^j} \left(\frac{\partial D_i}{\partial u^i} \right) + \frac{\partial^2}{\partial^2 u^i} \left(\frac{\partial D_j}{\partial u^j} \right)$$

$$= \sum_{l=1}^n \frac{\partial}{\partial u^i} \left(\frac{\partial}{\partial u^j} \left(g \left(R \left(u, \frac{\partial}{\partial x^i} \Big|_x \right) \frac{\partial}{\partial x^l} \Big|_x, R \left(u, \frac{\partial}{\partial x^j} \Big|_x \right) \frac{\partial}{\partial x^l} \Big|_x \right) \right),$$

which gives

$$2 \sum_{l=1}^n \left\| R \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^l} \right\|^2 = - \sum_{l=1}^n \left\| R \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^l} \right\|^2$$

Hence the Riemannian curvature R is zero, i.e. (M, g) is flat. We deduce that (TM, G) is flat and Z is a homothetic vector field, with homothety factor $\bar{\lambda}$.

The converse is easy to prove. \square

Remark 4.2. Note that for a Riemannian g -natural metric of the form $G = ag^s + bg^h + cg^v$ on TM , the flatness of (M, g) is equivalent to that of (TM, G) (cf. [10, Theorem 1.4]).

To classify completely Ricci solitons for $G = ag^s + bg^h + cg^v$, $a \neq 0$, $a + c \neq 0$ and $a(a + c) - b^2 \neq 0$, it is sufficient to characterize homothetic vector field on (TM, G) . We start with the following lemma which gives a local characterization of the Lie derivative with respect to an arbitrary vector field on TM :

Lemma 4.3. *Let (M, g) be a flat Riemannian manifold and $G = ag^s + bg^h + cg^v$. Then for all $Z \in \mathfrak{X}(TM)$ and $X, Y \in \mathfrak{X}(M)$, we have*

$$\begin{aligned} \mathcal{L}_Z G \left(X^h, Y^h \right) &= [(a + c) A^i + bB^i] \mathcal{L}_{\frac{\partial}{\partial x^i}} g(X, Y) \\ &\quad + \left[(a + c) X^h(A^i) + bX^h(B^i) \right] g \left(\frac{\partial}{\partial x^i}, Y \right) \\ &\quad + \left[(a + c) Y^h(A^i) + bY^h(B^i) \right] g \left(X, \frac{\partial}{\partial x^i} \right), \\ \mathcal{L}_Z G \left(X^h, Y^v \right) &= [b^i A^i + aB^i] g \left(\nabla_{\frac{\partial}{\partial x^i}} X, Y \right) \\ &\quad + \left[bX^h(A^i) + aX^h(B^i) \right] g \left(\frac{\partial}{\partial x^i}, Y \right) \\ &\quad + \left[(a + c) Y^v(A^i) + bY^v(B^i) \right] g \left(X, \frac{\partial}{\partial x^i} \right), \\ \mathcal{L}_Z G \left(X^v, Y^v \right) &= [bX^v(A^i) + aX^v(B^i)] g \left(\frac{\partial}{\partial x^i}, Y \right) \\ &\quad + [bY^v(A^i) + aY^v(B^i)] g \left(X, \frac{\partial}{\partial x^i} \right), \end{aligned}$$

where $Z = A^i \left(\frac{\partial}{\partial x^i} \right)^h + B^i \left(\frac{\partial}{\partial x^i} \right)^v$.

For a vector field ξ on M , we denote by $C(\xi)$ the $(1, 1)$ -tensor field on M defined by $g(C(\xi)Y, Z) = -g(Y, \nabla_Z \xi)$ for all vector fields Y and Z on M . Locally, if $\xi = \xi^i \frac{\partial}{\partial x^i}$, then $C(\xi)_j^i = -g^{ik} g_{jl} \xi_{;k}^l$, where $\xi_{;k}^l$ are the local components of the $(1, 1)$ -tensor field $\nabla \xi$. We denote also by I the identity $(1, 1)$ -tensor field on M . Then we have

Theorem 4.4. *Let (TM, G) be the tangent bundle of a flat Riemannian manifold (M, g) with $G = ag^s + bg^h + cg^v$, and Z be a vector field on TM . Z is a homothetic vector field on (TM, G) if and only if Z is expressed as*

$$Z = \frac{1}{\alpha} \left\{ [a\zeta - b\xi]^h + *[aC(\xi) - bP + \lambda abI] \right. \\ \left. + [(a+c)\xi - b\zeta]^v + \iota [(a+c)P - bC(\xi) + \lambda(\alpha - b^2)I] \right\}, \quad (4.4)$$

where

1. ζ is a homothetic vector field on M satisfying $\mathcal{L}_\zeta g = 2\lambda(a+c)g$;
2. P is a $(1,1)$ tensor field on M which is skew-symmetric and parallel;
3. ξ is a vector field on (M, g) satisfying $\nabla^2 \xi(U, V) + \nabla^2 \xi(V, U) = 0$ for any $U, V \in \mathfrak{X}(M)$.

Proof. Suppose that Z is a homothetic vector field, then there is a constant λ such that

$$\mathcal{L}_Z G = 2\lambda G. \quad (4.5)$$

Taking, in the third equation of Lemma 4.3, $X = \frac{\partial}{\partial x^i}$ and $Y = \frac{\partial}{\partial x^j}$ and taking into account (4.5), we obtain

$$2\lambda a g_{ij} = b \left[\frac{\partial A_i}{\partial u^j} + \frac{\partial A_j}{\partial u^i} \right] + a \left[\frac{\partial B_i}{\partial u^j} + \frac{\partial B_j}{\partial u^i} \right], \quad i, j = 1, \dots, n, \quad (4.6)$$

where $A_i := g_{ik}A^k$ and $B_i := g_{ik}B^k$. Putting

$$W_i := bA_i + aB_i - \lambda a u_i, \quad (4.7)$$

where $u_i := g_{ik}u^k$, it is easy to see that (4.6) is equivalent to

$$\frac{\partial W_i}{\partial u^j} + \frac{\partial W_j}{\partial u^i} = 0, \quad i, j = 1, \dots, n. \quad (4.8)$$

For $i = j$, (4.8) gives

$$\frac{\partial W_i}{\partial u^i} = 0, \quad i = 1, \dots, n. \quad (4.9)$$

Deriving (4.8) with respect to u^j and taking into account (4.9), we obtain

$$\frac{\partial^2 w_i}{(u^j)^2} = 0 \quad \text{for all } i, j = 1, \dots, n.$$

It follows that there are functions ξ_i, P_{ij} , $i, j = 1, \dots, n$, such that

$$W_i = \xi_i + P_{ik}u^k, \quad i = 1, \dots, n. \quad (4.10)$$

Equation (4.8) is equivalent to

$$P_{ij} + P_{ji} = 0, \quad i, j = 1, \dots, n. \quad (4.11)$$

Now we put

$$V^i := (a + c) A^i + bB^i, \quad i = 1, \dots, n. \quad (4.12)$$

Taking, in the second equation of Lemma 4.3, $X = \frac{\partial}{\partial x^i}$ and $Y = \frac{\partial}{\partial x^j}$ and taking into account (4.5), (4.10) and (4.12), we obtain

$$2\lambda b g_{ij} = g_{jk} \xi_{;i}^k + g_{jk} P_{l;i}^k u^l + g_{ik} \frac{\partial V^k}{\partial u^j}, \quad i, j = 1, \dots, n, \quad (4.13)$$

where $\xi^i := g^{ik} \xi_k$, $P_j^i := g^{ik} P_{kj}$ and $\xi_{;j}^i$ and $P_{j;k}^i$ are the local components of the differentials $\nabla \xi$ and ∇P of the local vector field $\xi := \xi^i \frac{\partial}{\partial x^i}$ and the local (1,1)-tensor field $P := P_j^i \frac{\partial}{\partial x^i} \otimes dx^j$, respectively, given by

$$\xi_{;j}^i = \frac{\partial \xi^i}{\partial x^j} + \Gamma_{jk}^i \xi^k, \quad P_{j;k}^i = \frac{\partial P_j^i}{\partial x^k} + \Gamma_{kl}^i P_j^l - \Gamma_{kj}^l P_l^i.$$

It follows then, from (4.13), that

$$\frac{\partial V^i}{\partial u^j} = 2\lambda b \delta_j^i - g^{il} g_{jk} \xi_{;l}^k - g^{im} g_{jk} P_{l;m}^k u^l, \quad i, j = 1, \dots, n. \quad (4.14)$$

We deduce that

$$V^i = \zeta^i + 2\lambda b u^i - g^{il} g_{jk} \xi_{;i}^k u^j + T_{jk}^i u^j u^k, \quad (4.15)$$

where ζ^i and T_{jk}^i are C^∞ -functions such that T_{jk}^i are symmetric in j and k . Substituting from (4.15) into (4.14), we obtain

$$2T_{jk}^i = -g^{im} g_{jl} P_{k;m}^l.$$

But we deduce from (4.11) that $-g^{im} g_{jl} P_{k;m}^l$ is skew-symmetric, and hence

$$P_{j;k}^i = 0, \quad i, j, k = 1, \dots, n. \quad (4.16)$$

It follows then that

$$V^i = \zeta^i + 2\lambda b u^i - g^{il} g_{jk} \xi_{;i}^k u^j, \quad i = 1, \dots, n,$$

or, in other words,

$$V^i = \zeta^i + 2\lambda b u^i + C(\xi)_j^i u^j, \quad i = 1, \dots, n, \quad (4.17)$$

where $C(\xi)$ is the (1,1)-tensor field given in Theorem (4.4), given locally by $C(\xi)_j^i = -g^{il} g_{jk} \xi_{;l}^k$.

Finally, taking, in the first equation of Lemma 4.3, $X = \frac{\partial}{\partial x^i}$ and $Y = \frac{\partial}{\partial x^j}$ and taking into account (4.5) and (4.17), we obtain

$$\begin{aligned} 2\lambda(a+c)g_{ij} &= g_{ik} \zeta_j^k + g_{jk} \zeta_i^k + \left(g_{ik} C(\xi)_{l;j}^k + g_{jk} C(\xi)_{l;i}^k \right) u^l \\ &= g_{ik} \zeta_{;j}^k + g_{jk} \zeta_{;i}^k - g_{lm} (\xi_{;ij}^m + \xi_{;ji}^m) u^l, \quad i, j = 1, \dots, n. \end{aligned} \quad (4.18)$$

or, in other words

$$\mathcal{L}_V \tilde{G} = 2a[\lambda - \mu]\tilde{g}^s + 2[a(\mu - \nu) + c(\lambda - \nu)]\tilde{g}^v + 2[\lambda d - \theta]\tilde{k}^v,$$

where \tilde{g}^s , \tilde{g}^v and \tilde{k}^v are the induced metrics on T_1M from g^s , g^v and k^v , respectively.

Remark 5.2. With the same hypotheses of Proposition 5.1, if (\tilde{G}, V, λ) is a Ricci soliton then

- V is a conformal vector field on (T_1M, \tilde{G}) if and only if $\theta = \mu d$ and $\mu = \nu$,
- V is a Killing vector field on (T_1M, \tilde{G}) if and only if $\theta = \lambda d$ and $\lambda = \mu = \nu$.

5.1. Case when the potential vector field is a complete lift vector field Recall that if ξ is a vector field on M , then its complete lift $\xi^{\bar{c}}$ to T_1M can be expressed as $\xi^{\bar{c}} = X^h + [\iota(\nabla\xi)]^t$. Using Lemmas 2.4 and 2.5, we have

Lemma 5.3 ([1]). *For all $\xi \in \mathfrak{X}(M)$, $(x, u) \in T_1M$ and $X, Y \in T_xM$, we have*

$$\begin{aligned} \left(\mathcal{L}_{\xi^{\bar{c}}}\tilde{G}\right)_{(x,u)}(X^h, Y^h) &= (a+c)(\mathcal{L}_{\xi g})_x(X, Y) + d\{(\mathcal{L}_{\xi g})_x(Y, u)g(X, u) \\ &\quad + (\mathcal{L}_{\xi g})_x(X, u)g(Y, u) - (\mathcal{L}_{\xi g})_x(u, u)g(Y, u)g(X, u)\}, \\ \left(\mathcal{L}_{\xi^{\bar{c}}}\tilde{G}\right)_{(x,u)}(X^h, Y^t) &= ag(R(\xi_x, X)u + \nabla^2\xi(u, X), Y), \\ \left(\mathcal{L}_{\xi^{\bar{c}}}\tilde{G}\right)_{(x,u)}(X^t, Y^t) &= a[(\mathcal{L}_{\xi g})_x(X, Y) - (\mathcal{L}_{\xi g})_x(u, u)g(X, Y)]. \end{aligned}$$

Theorem 5.4. *Let (M, g) be a Riemannian manifold of constant sectional curvature κ , \tilde{G} be a pseudo-Riemannian g -natural metric of Kaluza–Klein type on T_1M , ξ be a vector field on M and $\lambda \in \mathbb{R}$. Then, $(\tilde{G}, \xi^{\bar{c}}, \lambda)$ is a Ricci soliton if and only if the two following assertions hold*

1. ξ is a homothetic vector field on M , with $\mathcal{L}_{\xi}g = 2\lambda_0g$, where $\lambda_0 \in \mathbb{R}$;
2. one of the following cases occurs

$$\text{i) } n = 2, d = 0, \lambda = \frac{a\kappa^2}{2(a+c)^2} \text{ and } \lambda_0 = \kappa \frac{a\kappa - (a+c)}{(a+c)^2};$$

$$\text{ii) } n = 2, d \neq 0, \kappa = \frac{2\varphi - d}{a} \text{ and } \lambda = \lambda_0 = \frac{2}{a};$$

$$\text{iii) } n \neq 2, d = 0, \kappa = 0 \text{ and } \lambda_0 = \lambda = \frac{n-2}{a};$$

$$\text{iv) } n \neq 2, d \neq 0, \kappa = -\frac{1}{a(n-2)}[(n-1)d \pm \sqrt{d^2 + 2n(n-2)\varphi d}], \lambda = \mu \\ \text{and } \lambda_0 = \mu - \nu, \text{ where } \mu \text{ and } \nu \text{ are given by (5.2).}$$

Proof. Suppose that $(\tilde{G}, \xi^{\tilde{e}}, \lambda)$ is a Ricci soliton. For all $(x, u) \in TM$ and $X, Y \in M_x$. Substituting from the first equation of Lemma 5.3 into the first equation of (5.1) and taking $X = Y \perp u$, we obtain

$$g(\nabla_X \xi, X) = (\lambda - \nu)g(X, X). \quad (5.3)$$

Then ξ is a homothetic vector field with $\mathcal{L}_\xi g = 2(\lambda - \nu)g$. Substituting from the third equation of Lemma 5.3 into (5.1), we obtain

$$\lambda = \mu = \frac{1}{2a(a+c)\varphi} [a^2\kappa^2 + 2(n-2)(a+c)\varphi - d^2]. \quad (5.4)$$

We deduce that ξ is a homothetic vector field, i.e. $\mathcal{L}_\xi g = 2\lambda_0 g$, with

$$\lambda_0 = \mu - \nu. \quad (5.5)$$

Substituting from the first equation of Lemma 5.3 into the first equation of (5.1), we obtain

$$\nu d = \theta. \quad (5.6)$$

Thus

$$(n-2)a^2\varphi\kappa^2 + 2(n-1)ad\varphi\kappa + d[d^2 - (n+1)d\varphi + (a+c)(d-2n\varphi)] = 0,$$

i.e.,

$$(n-2)a^2\kappa^2 + 2(n-1)ad\kappa - nd(2\varphi - d) = 0. \quad (5.7)$$

Then we can distinguish the following situations:

i) $n = 2$ and $d = 0$: substituting into (5.2), we get $\mu = \frac{a}{2} \left(\frac{\kappa}{a+c} \right)^2$ and $\nu = \frac{\kappa}{2(a+c)^2} [-a\kappa + 2(a+c)]$. Then we have, by virtue of (5.4) and (5.5), $\lambda = \frac{a\kappa}{2(a+c)^2}$ and $\lambda_0 = \frac{\kappa}{(a+c)^2} [a\kappa - (a+c)]$.

ii) $n = 2$ and $d \neq 0$: (5.7) yields $\kappa = \frac{2\varphi-d}{a}$. Substituting into (5.2), we get $\mu = \frac{2}{a}$ and $\nu = 0$. Replacing in (5.4) and (5.5), we obtain $\lambda = \lambda_0 = \frac{2}{a}$.

iii) $n \neq 2$ and $d = 0$: then we have, from (5.6), $\theta = 0$. Using (5.2), we obtain $\kappa = 0$ and thus substituting into (5.2), we get $\mu = (n-2)\frac{a+c}{a\varphi}$ and $\nu = 0$. Replacing in (5.4) and (5.5), we obtain $\lambda = \lambda_0 = \frac{n-2}{a}$.

iv) $n \neq 2$ and $d \neq 0$: from equation (5.7) we obtain

$$\left[\kappa + \frac{(n-1)d}{(n-2)a} \right]^2 = \frac{d}{(n-2)^2 a^2} [d^2 + 2n(n-2)\varphi d]. \quad (5.8)$$

We deduce that $\kappa = -\frac{1}{a(n-2)} [(n-1)d \pm \sqrt{d^2 + 2n(n-2)\varphi d}]$.

Conversely, by a routine calculation, we can check that in any case **i)**–**iv)** of 2 of the theorem, we get

$$\lambda = \mu, \quad \lambda - \nu = \lambda_0, \quad \lambda d - \theta = \lambda_0 d. \quad (5.9)$$

On the other hand, it is well known that, for any homothetic vector field ξ on M , we have

$$R(\xi, X)Y + \nabla^2 \xi(Y, X) = 0 \quad (5.10)$$

for any $X, Y \in \mathfrak{X}(M)$. Substituting from the identity $\mathcal{L}_\xi g = 2\lambda_0 g$ and (5.10) into Lemma 5.3, and using (5.9), we obtain

$$\begin{aligned} (\mathcal{L}_{\xi^{\tilde{G}}} \tilde{G})_{(x,u)}(X^h, Y^h) &= 2\lambda_0(a+c)g(X, Y) + 2\lambda_0 dg(Y, u)g(X, u) \\ &= 2(a+c)(\lambda - \nu)g(X, Y) + 2(\lambda d - \theta)g(X, u)g(Y, u), \\ (\mathcal{L}_{\xi^{\tilde{G}}} \tilde{G})_{(x,u)}(X^h, Y^t) &= 0, \\ (\mathcal{L}_{\xi^{\tilde{G}}} \tilde{G})_{(x,u)}(X^t, Y^t) &= 0 = 2a[\lambda - \mu]g(X, Y), \end{aligned}$$

which is no other than (5.1), i.e. $(\tilde{G}, \xi^{\tilde{G}}, \lambda)$ is a Ricci soliton. \square

Remarks 5.5. Let (M, g) be a Riemannian manifold of constant sectional curvature κ , \tilde{G} be a pseudo-Riemannian g -natural metric on T_1M of Kaluza-Klein type, ξ be a vector field on M and $\lambda \in \mathbb{R}$.

1. (T_1M, \tilde{G}) is an Einstein manifold if and only if ξ is a Killing vector field on M , i.e. $\lambda_0 = 0$.
2. For $n = 2$ and $d = 0$, $(\tilde{G}, \xi^{\tilde{G}}, \lambda)$ is shrinking, steady or expanding according to $a < 0$, $\kappa = 0$ or $a > 0$, respectively.
3. For $n = 2$ and $d \neq 0$, $(\tilde{G}, \xi^{\tilde{G}}, \lambda)$ is shrinking or expanding according to $a < 0$ or $a > 0$, respectively.
4. For $n \neq 2$ and $d = 0$, $(\tilde{G}, \xi^{\tilde{G}}, \lambda)$ is shrinking or expanding according to $a < 0$ or $a > 0$, respectively.
5. For $n \neq 2$ and $d \neq 0$, $(\tilde{G}, \xi^{\tilde{G}}, \lambda)$ is shrinking, steady, or expanding according to μ is negative, zero or positive, respectively.
6. In the case 2 iv) of Theorem 5.4, we should have $d^2 + 2n(n-2)\varphi d \geq 0$, i.e. one of the following situations occurs
 - $a + c < 0$ and $d \in]-\infty, 0] \cup [\frac{-2n(n-2)(a+c)}{2n(n-2)+1}, +\infty[$,
 - $a + c > 0$ and $d \in]-\infty, \frac{-2n(n-2)(a+c)}{2n(n-2)+1}] \cup [0, +\infty[$.

5.2. Necessary conditions on potential vector fields. Since T_1M is a closed submanifold of TM , then for any vector field V on T_1M , there is a vector field \bar{V} on TM which extends V . For a local coordinates system (U, x^1, \dots, x^n) of M , \bar{V} can be expressed in the induced coordinates system of TM as

$$\bar{V} \upharpoonright_{p^{-1}(U)} = A^i \left(\frac{\partial}{\partial x^i} \right)^h + B^i \left(\frac{\partial}{\partial x^i} \right)^v,$$

where A^i and B^i are smooth functions on $p^{-1}(U)$. Since \bar{V} is tangent to T_1M at any point of T_1M , we should have $g_{ij}(x) B^i(x, u) u^j = 0$ for all $(x, u) \in p_1^{-1}(U) = p^{-1}(U) \cap T_1M$ expressed locally as $u = u^i \frac{\partial}{\partial x^i}$, i.e.,

$$B^i(x, u) u_i = 0, \quad (5.11)$$

where we put $u_i := g_{ij}(x) u^j$. If we denote the restrictions to $p_1^{-1}(U)$ of A^i and B^i by the same notations, then we can write

$$V \upharpoonright_{p_1^{-1}(U)} = A^i \left(\frac{\partial}{\partial x^i} \right)^h + B^i \left(\frac{\partial}{\partial x^i} \right)^t. \quad (5.12)$$

Using Lemmas 2.4 and 2.5, we can prove the following lemma.

Lemma 5.6. *For all $(x, u) \in T_1M$, $X, Y \in M_x$, we have*

$$\begin{aligned} (\mathcal{L}_V \tilde{G})_{(x,u)}(X^h, Y^h) &= (a+c) \left[A^i \left(\mathcal{L}_{\frac{\partial}{\partial x^i}} g \right)_x(X, Y) \right. \\ &\quad \left. + X^h(A^i) g \left(\frac{\partial}{\partial x^i} \Big|_x, Y \right) + Y^h(A^i) g \left(X, \frac{\partial}{\partial x^i} \Big|_x \right) \right] \\ &\quad + d \left\{ B^i \left[g \left(X, \frac{\partial}{\partial x^i} \Big|_x \right) g(Y, u) + g \left(\frac{\partial}{\partial x^i} \Big|_x, Y \right) g(X, u) \right] \right. \\ &\quad \left. + A^i \left[g \left(\nabla_X \frac{\partial}{\partial x^i}, u \right) g(Y, u) + g \left(\nabla_Y \frac{\partial}{\partial x^i}, u \right) g(X, u) \right] \right. \\ &\quad \left. + X^h(A^i) g \left(\frac{\partial}{\partial x^i} \Big|_x, u \right) g(Y, u) + Y^h(A^i) g \left(\frac{\partial}{\partial x^i} \Big|_x, u \right) g(X, u) \right\}, \\ (\mathcal{L}_V \tilde{G})_{(x,u)}(X^h, Y^t) &= a \left[X^h(B^i) g \left(\frac{\partial}{\partial x^i} \Big|_x, Y \right) + B^i g \left(\nabla_X \frac{\partial}{\partial x^i}, Y \right) \right. \\ &\quad \left. + k A^i \left(g(X, u) g \left(\frac{\partial}{\partial x^i} \Big|_x, Y \right) - g(X, Y) g \left(\frac{\partial}{\partial x^i} \Big|_x, u \right) \right) \right] \\ &\quad + (a+c) Y^t(A^i) g \left(X, \frac{\partial}{\partial x^i} \Big|_x \right) + d Y^t(A^i) g(X, u) g \left(\frac{\partial}{\partial x^i} \Big|_x, u \right), \\ (\mathcal{L}_V \tilde{G})_{(x,u)}(X^t, Y^t) &= a \left[Y^t(B^i) g \left(X, \frac{\partial}{\partial x^i} \Big|_x \right) + X^t(B^i) g \left(Y, \frac{\partial}{\partial x^i} \Big|_x \right) \right]. \end{aligned}$$

We put

$$W = V - A^l \left(\frac{\partial}{\partial x^l} \right)^h.$$

W is a fiber-preserving vector field, and we have, from Lemma 5.6,

$$\mathcal{L}_W \tilde{G}(X^t, Y^t) = \mathcal{L}_V \tilde{G}(X^t, Y^t).$$

Using (5.1), we deduce that

$$\mathcal{L}_W \tilde{G}(X^t, Y^t) = 2[\lambda - \mu] \tilde{G}(X^t, Y^t),$$

In the same way, we remark that

$$\mathcal{L}_W \tilde{g}^s(X^t, Y^t) = 2[\lambda - \mu] \tilde{g}^s(X^t, Y^t). \quad (5.13)$$

We shall extend fiber-preserving vector fields on T_1M to vector fields on TM , using the techniques used in [22]. Let Z be a fiber-preserving vector field on T_1M . Then it is known that Z is projectable to a vector field \underline{Z} on M , i.e., such that $(dp_1)_u(Z_u) = \underline{Z}_x$ for all $x \in M$ and $u \in S_xM$. For all $r > 0$, let us define the immersions $j_r : T_1M \rightarrow TM$ and $j_0 : M \rightarrow TM$, respectively, by $j_r(u) = ru$ for all $u \in T_1M$, and $j_0(x) = 0_x$ for all $x \in M$, where 0_x denotes the zero vector in M_x . We define a vector field $\overline{\overline{Z}}$ on TM extending Z by (cf. [22])

$$\overline{\overline{Z}}_{ru} := \begin{cases} (dj_r)_u(Z_u) & \text{for } r > 0 \\ (dj_0)_x(\underline{Z}_x) & \text{for } r = 0 \end{cases}$$

for all $x \in M$ and $u \in S_xM := M_x \cup T_1M$. We denote by $\overline{\overline{Z}}$ the restriction of $\overline{\overline{Z}}$ to $TM \setminus \sigma_0$, which is clearly a vector field on $TM \setminus \sigma_0$, where $\sigma_0 := j_0(M)$ is the zero section of TM .

Lemma 5.7 ([1]). *Let Z, X and Y be fiber-preserving vector fields on T_1M . Then*

$$(\mathcal{L}_{\overline{\overline{Z}}} g^s(\overline{X}, \overline{Y}))_{ru} = (1 - r^2)(a + c)(\mathcal{L}_{\underline{Z}} g(\underline{X}, \underline{Y}))_x + r^2(\mathcal{L}_Z \tilde{g}^s(X, Y))_u \quad (5.14)$$

for all $x \in M$, $u \in S_xM$ and $r > 0$.

Now, fixing $u \in S_xM$ and taking, in (5.14), $Z = \overline{\overline{W}}$ and $X = Y = T^t$ for $T \in \mathfrak{X}(M)$ such that $0 \neq T_x \perp u$, and using Lemma 5.7, we obtain

$$\mathcal{L}_{\overline{\overline{W}}_{ru}} g^s(T_{ru}^v, T_{ru}^v) = \mathcal{L}_{Z_u} \tilde{g}^s(T_u^v, T_u^v).$$

Since, by (5.13), we have $\mathcal{L}_{Z_u} \tilde{g}^s(T_u^v, T_u^v) = 2[\lambda - \mu]g(T_x, T_x)$, then

$$\mathcal{L}_{\overline{\overline{W}}_{ru}} g^s(T_{ru}^v, T_{ru}^v) = 2[\lambda - \mu]g(T_x, T_x)$$

for all $r > 0$. When $r \rightarrow 0$, we have by continuity

$$\mathcal{L}_{\overline{\overline{Z}}_0} g^s(T_0^v, T_0^v) = 2[\lambda - \mu]g(T_x, T_x).$$

But since $\overline{\overline{Z}}_0 = 0$, then $2[\lambda - \mu]g(T_x, T_x) = 0$, and consequently

$$\lambda = \mu = \frac{1}{2a(a+c)\varphi} [a^2\kappa^2 + 2(n-2)(a+c)\varphi - d^2]$$

and $\mathcal{L}_{\bar{Z}}g^s(T_{ru}^v, T_{ru}^v) = 0$. Using the third identity of Lemma 4.3 for $G = g^s$, we obtain easily, as in the proof of Theorem 4.4,

$$B^i = Q_j^i u^j + \zeta^i, \quad i = 1, \dots, n, \quad (5.15)$$

where Q_j^i, ζ^i are smooth functions such that

$$g_{ik}Q_j^k + g_{jk}Q_i^k = 0, \quad i, j = 1, \dots, n. \quad (5.16)$$

Moreover since $B^i u_i = 0$, then $Q_j^i u_j u^i + \zeta^i u_i = 0$. Replacing u by $-u$ in the last identity, we deduce that $\zeta^i = 0$ for all $i = 1, \dots, n$, and (5.15) becomes $B^i = Q_j^i u^j$. Considering the $(1, 1)$ -tensor field Q on M whose components are Q_j^i , then we have $W_{(x,u)} = Q(u)$ for all $(x, u) \in T_1M$, or equivalently $t\{V\} = \tilde{\iota}(Q)$, with Q is skew-symmetric. This completes the proof of the following proposition.

Proposition 5.8. *Let (M, g) be a Riemannian manifold of constant sectional curvature κ , \tilde{G} be a pseudo-Riemannian g -natural metric on T_1M of Kaluza–Klein type, V be a vector field on T_1M and $\lambda \in \mathbb{R}$. If (V, \tilde{G}, λ) is a Ricci soliton on T_1M then we have*

1. $\lambda = \frac{1}{2a(a+c)}[a^2\kappa^2 + 2(n-2)(a+c)\varphi - d^2]$;
2. $t\{V\} = \tilde{\iota}(Q)$, where Q is a skew-symmetric $(1, 1)$ -tensor field.

5.3. Case when the potential vector field is fiber-preserving. In this section, we shall give necessary and sufficient conditions for the unit tangent bundle of a constant curvature Riemannian manifold, endowed with a pseudo-Riemannian Kaluza–Klein type metric, to be a Ricci soliton with a fiber preserving potential vector field.

Theorem 5.9. *Let (M, g) be an n -dimensional Riemannian manifold of constant sectional curvature κ , \tilde{G} be a pseudo-Riemannian g -natural metric on T_1M of Kaluza–Klein type, V be a fiber-preserving vector field on T_1M and $\lambda \in \mathbb{R}$. Then (\tilde{G}, V, λ) is a Ricci soliton on T_1M if and only if there are:*

1. A conformal vector field ξ on M , with $\mathcal{L}_\xi g = 2\lambda_0 g$;
2. A parallel $(1, 1)$ -tensor field P on M , with $P - (\lambda - \nu)I$ is skew-symmetric; such that one of the following cases occurs:
 - i) $n = 2, d = 0, \lambda = \frac{ak^2}{2(a+c)^2}, \lambda_0 = k \frac{ak - (a+c)}{(a+c)^2}$ and $V = \xi^{\bar{c}} + \tilde{i}P$;
 - ii) $n = 2, d \neq 0, k = \frac{2\varphi - d}{a}, \lambda = \lambda_0 = \frac{2}{a}$ and $V = \xi^{\bar{c}}$;
 - iii) $n \neq 2, d = 0, k = 0, \lambda_0 = \lambda = \frac{n-2}{a}$ and $V = \xi^{\bar{c}} + \tilde{i}P$;

iv) $n \neq 2, d \neq 0, k = -\frac{1}{a(n-2)}[(n-1)d \pm \sqrt{d^2 + 2n(n-2)\varphi d}], \lambda = \mu, \lambda_0 = \mu - \nu$ and $V = \xi^{\bar{c}}$;

where I is the identity $(1,1)$ -tensor field on M and μ and ν are given by (5.2).

Proof. Let V be a fiber-preserving vector field on T_1M . Then

$$h\{[V, X^t](x, u)\} = 0 \quad (5.17)$$

for all $(x, u) \in T_1M$ and $X \in M_x$, where h stands for horizontal component. Using the identity $B^i(x, u)u_i = 0$, we obtain $h\{[V, X^t](x, u)\} = h\{-X^t(A^l)\frac{\partial}{\partial t}\}$. Then $h\{[V, X^t](x, u)\} = 0$ if and only if

$$X^t(A^l) = 0$$

for all $(x, u) \in T_1M$ and $X \in M_x$. It follows that, for $X \perp u$, we have $(dA^l)_{(x,u)}(X^t_{(x,u)}) = 0$ for all $X \perp u$. If we denote by A^l_x the restriction of A^l to the fiber $S_xM := T_1M \cap M_x$, we then have

$$(dA^l_x)_{(x,u)}(X^t_{(x,u)}) = (dA^l)_{(x,u)}(X^t) = 0 \quad (5.18)$$

for all $X \in M_x$ such that $g(X, u) = 0$. But dA^l_x is a linear form defined on the tangent space

$$(S_xM)_{(x,u)} = \{X^t_{(x,u)} / X \in M_x, g(X, u) = 0\}.$$

Thus, by (5.18) it follows that $(dA^l_x)_{(x,u)}$ vanishes identically on $(S_xM)_{(x,u)}$. We conclude that the restriction A^l_x is constant on S_xM . Therefore, for any $x \in M$, there is a C^∞ -function ξ^l on M , such that

$$A^l_{(x,u)} = \xi^l(x).$$

i.e., $\pi_*(V) = \xi$, where ξ is the vector field on M given locally by $\xi = \sum_{i=1}^n \xi^i \left(\frac{\partial}{\partial x^i}\right)$. It follows then that $h\{V\} = \xi^h$. But, by Proposition 5.8, we have $t\{V\} = \tilde{t}Q$ for some skew symmetric $(1,1)$ -tensor field Q on M . We deduce that V is given by $V = \xi^h + \tilde{t}Q$. It is easy to see from Lemma 5.6 that, for all $(x, u) \in T_1M, X, Y \in M_x$, we have

$$\left\{ \begin{array}{l} (\mathcal{L}_V \tilde{G})_{(x,u)}(X^h, Y^h) = (a+c)(\mathcal{L}_\xi g)_x(X, Y) \\ \quad + d[g(\nabla_X \xi, u)g(Y, u) + g(\nabla_Y \xi, u)g(X, u) \\ \quad + g(Q(u), X)g(Y, u) + g(Q(u), Y)g(X, u)]; \\ (\mathcal{L}_V \tilde{G})_{(x,u)}(X^h, Y^t) = ag(R(\xi_x, X)u + \nabla_X Q(u), Y); \\ (\mathcal{L}_V \tilde{G})_{(x,u)}(X^t, Y^t) = a[g(Q(X), Y) + g(Q(Y), X)]. \end{array} \right. \quad (5.19)$$

Putting, in the first equation of (5.19), $X = Y = u$ and using (5.1), we get

$$\mathcal{L}_\xi g(X, X) = 2(\lambda - \nu) \|X\|^2$$

for all $X \perp u$. Since u is arbitrary, the previous identity holds for all $X \in TM$. Then, by bilinearity, we get

$$\mathcal{L}_\xi g(X, Y) = 2(\lambda - \nu) g(X, Y) \quad (5.20)$$

for all $x \in M$ and $X, Y \in M_x$. Taking $X = u$ and $Y \perp u$ into the first equation of (5.19) we obtain, by virtue of (5.1) and (5.20),

$$d[g(Q(u), Y) + g(\nabla_Y \xi, u)] = 0 \quad (5.21)$$

for all $Y \perp u$. So, we have two possibilities:

Case 1: If $d \neq 0$, then from (5.21) we obtain $g(Q(u), Y) = -g(\nabla_Y \xi, u)$ for all $Y \perp u$. But we have, by (5.20), $g(\nabla_Y \xi, u) + g(\nabla_Y \xi, u) = 2(\lambda - \nu) g(Y, u) = 0$ for all $Y \perp u$. It follows that

$$g(Q(u) - \nabla_u \xi, Y) = 0 \quad (5.22)$$

for all $Y \perp u$. Now, since Q is skew-symmetric, then we have, by virtue of (5.20),

$$g(Q(u) - \nabla_u \xi, u) = -g(\nabla_u \xi, u) = (\nu - \lambda) \quad (5.23)$$

for all $(x, u) \in T_1 M$. Combining (5.22) and (5.23), we obtain $Q(u) = \nabla_u \xi + (\nu - \lambda)u$, and consequently $(iQ)_{(x,u)} = (\nabla_u \xi)_{(x,u)}^t$, since $u_{(x,u)}^t = 0$. We deduce then that $V_{(x,u)} = \xi_{(x,u)}^h + (\nabla_u \xi)^t = \xi_{(x,u)}^{\bar{c}}$, i.e.

$$V = \xi^{\bar{c}}. \quad (5.24)$$

Case 2: If $d = 0$, then we consider the $(1, 1)$ tensor field P given by $P(X) = Q(X) - \nabla_X \xi$ for all $X \in TM$. Then V can be expressed as $V = \xi^{\bar{c}} + \tilde{i}P$. Recall that, by (5.20), ξ is homothetic and hence (5.10) holds. Then the system (5.19) reduces to

$$\left\{ \begin{array}{l} (\mathcal{L}_V \tilde{G})_{(x,u)}(X^h, Y^h) = 2(a + c)(\lambda - \nu)g_x(X, Y); \\ (\mathcal{L}_V \tilde{G})_{(x,u)}(X^h, Y^t) = ag(\nabla_X P(u), Y); \\ (\mathcal{L}_V \tilde{G})_{(x,u)}(X^t, Y^t) = a[g(P(X), Y) + g(P(Y), X)]. \end{array} \right. \quad (5.25)$$

Comparing with (5.1), we obtain

$$\theta = 0, \quad \nabla P = 0, \quad \text{and} \quad g(P(X), Y) + g(P(Y), X) = 2(\lambda - \nu)g(X, Y).$$

Now $\theta = 0$ means by (5.2) that either $n = 2$ or $k = 0$.

The converse is straightforward. \square

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Природні солітони Річчі на дотичних та одиничних дотичних розшаруваннях

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Розглядаючи псевдоріманові g -природні метрики на дотичних розшаруваннях, ми доводимо, що умова бути солітоном Річчі є спадковою в тому сенсі, що структура солітона Річчі на дотичному розшаруванні породжує структуру солітона Річчі на базовому многовиді. Обмежуючись одним класом псевдоріманових g -природних метрик, ми покажемо, що дотичне розшарування є солітоном Річчі тоді і тільки тоді, коли базовий многовид є плоским, а потенціальне векторне поле є повним підйомом конформного векторного поля. Потім ми надаємо класифікацію конформних векторних полів на дотичному розшаруванні плоского ріманового многовиду, спорядженого згаданими g -природними метриками. Коли одиничні дотичні розшарування над рімановим многовидом з постійною кривиною наділені псевдорімановою метрикою типу Калуці–Клейна, ми даємо класифікацію структур солітонів Річчі, потенціальні векторні поля яких зберігають шари, роблячи висновок про існування таких структур, що не є ейнштейновими.

Ключові слова: дотичне розшарування, одиничне дотичне (сферичне) розшарування, g -природні метрики, метрики типу Калуці–Клейна, солітони Річчі