Journal of Mathematical Physics, Analysis, Geometry
2022, Vol. 18, No. 1, pp. 33–56
doi: https://doi.org/10.15407/mag18.01.033

Jacobi–Lie Hamiltonian Systems on Real Low-Dimensional Jacobi–Lie Groups and their Lie Symmetries

H. Amirzadeh-Fard, Gh. Haghighatdoost, and A. Rezaei-Aghdam

We study Jacobi–Lie Hamiltonian systems admitting Vessiot–Guldberg Lie algebras of Hamiltonian vector fields related to Jacobi structures on real low-dimensional Jacobi-Lie groups. Also, we find all possible examples of Jacobi–Lie Hamiltonian systems on real two- and three-dimensional Jacobi–Lie groups. Finally, we present Lie symmetries of Jacobi–Lie Hamiltonian systems on the real three-dimensional Lie group $SL(2, \mathbb{R})$.

Key words: Jacobi–Lie group, Jacobi manifold, Lie system, Jacobi–Lie Hamiltonian system, Lie symmetry

Mathematical Subject Classification 2010: 34L15, 34L20, 35R10

1. Introduction

A Lie system is a t-dependent system of first-order ODEs that possesses a superposition rule [5, 6, 16]. In other words, the Lie system amounts to a t-dependent vector field that takes values in a finite-dimensional real Lie algebra of vector fields, the so-called Vessiot–Guldberg Lie algebra (VG Lie algebra) of the system.

The analysis of Lie systems dates back to the end of the 19th century, when Lie, Vessiot, Guldberg, Köningsberger and others [9, 13, 16, 23] pioneered the study of systems of ODEs admitting superposition rules. For almost a century, the study of this problem was not considered. After the work of Winternitz [24], many authors investigated this problem [3–9, 13, 17, 23, 24]. Some results were obtained for Lie systems admitting a VG Lie algebra of Hamiltonian vector fields relative to symplectic and Poisson structures [7].

A particular class of Lie systems on Poisson manifolds, the so-called Lie–Hamilton systems, that admits a VG Lie algebra of Hamiltonian vector fields with respect to a Poisson structure was studied in [7]. Indeed, Lie–Hamilton systems are a generalization of t-dependent Hamiltonian systems. Lie systems possessing VG Lie algebras of Hamiltonian vector fields with respect to Jacobi structures [11,12,15], referred to as Jacobi–Lie systems, were studied and exactly introduced in [10]. In our previous work [2], we studied Jacobi–Lie Hamiltonian systems on real low-dimensional Lie groups.

It is well known that the symplectic manifold is a particular case of the Poisson manifold because the Poisson bracket is not necessarily assumed to be non-degenerate. Since the Jacobi bracket is not necessarily a derivation, the Jacobi manifold is a generalization of the Poisson manifold [12,15].

In this work, we study Lie systems with VG Lie algebras of Hamiltonian vector fields with respect to Jacobi–Lie groups [11] especially on real two- and three-dimensional Jacobi–Lie groups [20,21]. Moreover, we find all possible Jacobi–Lie Hamiltonian systems on real low-dimensional Jacobi-Lie groups and we present Lie symmetries for Jacobi–Lie Hamiltonian systems.

The outline of the paper is as follows: In Section 2, we recall several definitions and results on Lie and Lie–Hamilton systems, Jacobi structures on Jacobi–Lie groups [11], and Jacobi–Lie Hamiltonian systems [10]. In Section 3, we exemplify the results of Section 2 on real two- and three-dimensional Jacobi–Lie groups [20, 21] and find all possible examples. Finally, in Section 4, we study Lie symmetries of Jacobi–Lie Hamiltonian systems with good Hamiltonian functions on $\mathbb{VIII} = SL(2,\mathbb{R})$.

2. Definitions and notations on Lie and Jacobi–Lie Hamiltonian systems

Throughout the paper, we assume that all functions and geometric structures are real, smooth and globally defined. To highlight the main aspects of our results, let us omit minor technical problems. Here, for self-containment of the paper, we review some basic concepts of Lie, Lie–Hamilton [7,17] and Jacobi–Lie Hamiltonian systems [10].

2.1. Lie systems and Lie–Hamilton systems. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two vector subsets of Lie algebra $\mathfrak{g}$, and let $[\mathfrak{a},\mathfrak{b}]$ denote the vector space spanned by the Lie brackets between the elements of $\mathfrak{a}$ and $\mathfrak{b}$. We define $Lie(\mathfrak{a},\mathfrak{g},[\cdot,\cdot])$ to be the smallest Lie subalgebra of $(\mathfrak{g},[\cdot,\cdot])$ containing $\mathfrak{a}$ and represent it by $Lie(\mathfrak{a})$.

Definition 2.1. A t-dependent vector field on a manifold M is a map

$$X : \mathbb{R}\times M \to TM, \quad (t,x)\mapsto X(t,x),$$

satisfying $\tau_M \circ X = \pi_2$, where π_2 and τ_M are the projections from $\mathbb{R}\times M$ and TM onto M, respectively.

Using this definition, we can identify every t-dependent vector field with a family $\{X_t\}_{t\in\mathbb{R}}$ of vector fields $X_t : M \to TM, x \mapsto X_t(x) = X(t,x)$, and vice versa.

Definition 2.2. The minimal Lie algebra of X on a manifold M is the smallest real Lie algebra $\mathfrak{g}^X$ containing the vector fields $\{X_t\}_{t\in\mathbb{R}}$. In other words, $\mathfrak{g}^X = \mathrm{Lie}\left(\{X_t\}_{t\in\mathbb{R}}\right)$.

Definition 2.3 ([1])**.** An integral curve of X is an integral curve $\alpha : \mathbb{R} \to \mathbb{R} \times M$, $t \mapsto (t, x(t))$, of the suspension of X, i.e., for the vector field

$$\tilde{X} : \mathbb{R} \times M \to T(\mathbb{R} \times M) \simeq T\mathbb{R} \oplus TM, \quad (t, x) \mapsto \frac{\partial}{\partial t} + X(t, x),$$

we have $\frac{dx(t)}{dt} = X(t, x(t))$.

Note that we are identifying X with its associated system of differential equations $\frac{dx(t)}{dt} = X(t, x(t))$.

Definition 2.4 ([17])**.** A Lie system is a system X on a manifold M whose $\mathfrak{g}^X$ is finite dimensional.

Definition 2.5 ([16])**.** A superposition rule depending on n particular solutions for a system X on a manifold M is a function $\Psi : M^n \times M \to M$, $(x_{(1)}, \ldots, x_{(n)}; k) \mapsto x$, such that the general solution $x(t)$ of X can be written as

$$x(t) = \Psi(x_{(1)}(t), \ldots, x_{(n)}(t); k),$$

where $x_{(1)}(t), \ldots, x_{(n)}(t)$ is any generic collection of particular solutions to X and $k = (k_1, \ldots, k_m)$ is a point of M to be related to the initial conditions of X.

Theorem 2.6 (Lie–Scheffers [6, 16])**.** *A system X on a manifold M admits a superposition rule if and only if it can be written in the form*

$$X(t, x) = \sum_{i=1}^{r} b_i(t) X_i(x)$$

for a set $b_1(t), \ldots, b_r(t)$ of t-dependent functions and a family of vector fields $X_1, \ldots, X_r$ on M spanning an r-dimensional real Lie algebra: a VG Lie algebra of X. In other words, a system X on a manifold M possesses a superposition rule if and only if it is a Lie system.

Definition 2.7 ([22])**.** A manifold M endowed with a bivector field $P \in \Gamma(\bigwedge^2 TM)$ satisfying $[P, P] = 0$ is called a Poisson manifold (M, P), where $[\cdot, \cdot]$ is the Schouten–Nijenhius bracket and $\Gamma(\bigwedge^2 TM)$ is the space of sections of $\bigwedge^2 TM$ over M.

Note that for general p-vector fields $P = X_1 \wedge \cdots \wedge X_p$ and q-vector fields $Q = Y_1 \wedge \cdots \wedge Y_q$, the Schouten–Nijenhius bracket is given by $[P, Q] = [X_1 \wedge \cdots \wedge X_p, Y_1 \wedge \cdots \wedge Y_q] = \sum_{\mathrm{i=1}}^{\mathrm{p}} \sum_{\mathrm{j=1}}^{\mathrm{q}} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_p \wedge Y_1 \wedge \cdots \wedge \hat{Y}_j \wedge \cdots \wedge Y_q$, for all $X_i, Y_i \in \mathfrak{X}(M)$, where $[X_i, Y_j]$ denotes the Lie bracket of the two vector fields X_i, Y_j on M.

We call P a Poisson bivector of the Poisson manifold (M, P). The Poisson bivector P induces a bundle morphism $P^\# : T^*M \to TM$ such that

$$\beta(P^\# \alpha) = P(\alpha, \beta), \quad \alpha, \beta \in T^*M.$$

Definition 2.8. A vector field X on M with the Poisson bivector P is said to be a Hamiltonian vector field if it can be written as $X = P^{\#}(df)$, where f is a function on M called Hamiltonian function. Conversely, every function f is called a Hamiltonian function of a unique Hamiltonian vector field X_f.

Definition 2.9 ([7]). A Lie–Hamilton system on a Poisson manifold M is a Lie system X whose $\mathfrak{g}^X$ consists of Hamiltonian vector fields relative to a Poisson bivector P.

2.2. Jacobi and Jacobi–Lie structures. Jacobi manifolds were studied by Lichnerowicz and Kirillov [12, 15].

Definition 2.10. A Jacobi manifold is a triple (M, Λ, E), where Λ is a bivector field, i.e., $\Lambda \in \bigwedge^2 \mathfrak{X}(M)$, and E is a vector field on M, i.e., $E \in \mathfrak{X}(M)$, (called the Reeb vector field) such that

$$[\Lambda, \Lambda] = 2E \wedge \Lambda, \qquad L_E\Lambda = [E, \Lambda] = 0,$$

where $[\cdot, \cdot]$ is the Schouten–Nijenhuis bracket, and L_E is the Lie derivative relative to the vector field E.

The space $(C^\infty(M), \{\cdot, \cdot\}_{\Lambda,E})$ is a local Lie algebra in the sense of Kirillov [12] with the following Jacobi bracket:

$$\{f, g\}_{\Lambda,E} = \Lambda(df, dg) + fEg - gEf, \quad f, g \in C^\infty(M).$$

This Lie bracket is a Poisson bracket if and only if the vector field E identically vanishes.

Definition 2.11. A Jacobi–Lie bialgebra is a pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$, where $\mathfrak{g}$ is a finite-dimensional real Lie algebra with the Lie bracket $[\cdot, \cdot]^{\mathfrak{g}}$ and $\mathfrak{g}^*$ is a dual space of $\mathfrak{g}$ with the Lie bracket $[\cdot, \cdot]^{\mathfrak{g}^*}$; also, $\phi_0 \in \mathfrak{g}^*$ and $X_0 \in \mathfrak{g}$ are 1-cocycles on $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively, such that for all $X, Y \in \mathfrak{g}$ satisfy the following properties:

$$\begin{aligned} d_{*X_0}[X, Y]^{\mathfrak{g}} &= [X, d_{*X_0}Y]^{\mathfrak{g}}_{\phi_0} - [Y, d_{*X_0}X]^{\mathfrak{g}}_{\phi_0}, \\ \phi_0(X_0) &= 0, \\ i_{\phi_0}(d_*X) + [X_0, X]^{\mathfrak{g}} &= 0, \end{aligned}$$

where

$$i_{\phi_0} : \wedge^k \mathfrak{g} \to \wedge^{k-1} \mathfrak{g}, \quad P \mapsto i_{\phi_0}P.$$

Moreover, d_*, being the Chevalley–Eilenberg differential of $\mathfrak{g}^*$ acting on $\mathfrak{g}$ and d_{*X_0}, is its generalization such that we have

$$\forall X, Y \in \mathfrak{g} \quad d_{*X_o}Y = d_*Y + X_o \wedge Y.$$

In addition, $[\cdot, \cdot]^{\mathfrak{g}}_{\phi_0}$ is the ϕ_0-Schouten–Nijenhuis bracket with the following properties:

$\forall P \in \wedge^k \mathfrak{g} \; \forall P' \in \wedge^{k'} \mathfrak{g}$

$$[P, P']_{\phi_o} = [P, P'] + (-1)^{k+1}(k-1)P \wedge i_{\phi_o}P' - (k'-1)i_{\phi_o}P \wedge P', \tag{2.1}$$
$$[P, P']_{\phi_o} = (-1)^{kk'}[P', P]_{\phi_o}. \tag{2.2}$$

If $((\mathfrak{g}, \phi_o), (\mathfrak{g}^*, X_o))$ is a Jacobi–Lie bialgebra, then $((\mathfrak{g}^*, X_o), (\mathfrak{g}, \phi_o))$ is also a Jacobi–Lie bialgebra. For this case, we have d as the Chevalley–Eilenberg differential of $\mathfrak{g}$ acting on $\mathfrak{g}^*$ and it has the following $\phi_0 \in \mathfrak{g}^*$ generalization:

$$\forall \omega \in \wedge^k \mathfrak{g}^* \quad d_{\phi_o}\omega = d\omega + \phi_o \wedge \omega. \tag{2.3}$$

In the above definition, X_0 and ϕ_o are 1-cocycles on $\mathfrak{g}^*$ and $\mathfrak{g}$, respectively, i.e., we must have

$$d_* X_o = 0, \quad d\phi_o = 0. \tag{2.4}$$

Definition 2.12 ([11])**.** A bivector Λ on a connected Lie group $\mathbb{G}$ is said to be $\sigma-$multiplicative if Λ satisfies the relation

$$\Lambda(gh) = (R_h)_*(\Lambda(g)) + e^{-\sigma(g)}(L_g)_*(\Lambda(h)), \quad g,\, h, \in \mathbb{G},$$

where $\sigma : \mathbb{G} \to \mathbb{R}$ is a multiplicative function. Meanwhile, $(L_g)_*$ $\big($respectively, $(R_h)_*\big)$ is the pullback of the left (respectively, the right) invariant action of $\mathbb{G}$ on itself.

Definition 2.13 ([11])**.** A Jacobi–Lie group is a connected Lie group $\mathbb{G}$ together with a Jacobi structure (Λ, E) on $\mathbb{G}$ such that:

i) Λ is $\sigma-$multiplicative.

ii) E is a right invariant vector field, $E(\mathfrak{e}) = -X_0$ and $\Lambda^{\#}(d\sigma) = \tilde{X}_0 - e^{-\sigma}\bar{X}_0$,

where $E = -\tilde{X}_0$ and $\mathfrak{e}$ is the identity element of $\mathbb{G}$; in addition, $\tilde{X}$ and $\bar{X}$ are the right and the left invariant vector fields, respectively, such that $\tilde{X}_0(\mathfrak{e}) = \bar{X}_0(\mathfrak{e}) = X_0$.

Iglesias and Marrero proved in [11] that if $\mathbb{G}$ is a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$ and the pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is a Jacobi–Lie bialgebra, then $\mathbb{G}$ is a Jacobi–Lie group and it has a special Jacobi structure.

Theorem 2.14 ([11])**.** *Let the pair* $\big((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0)\big)$ *be a Jacobi–Lie bialgebra and let* $\mathbb{G}$ *be a connected and simply connected Lie group with Lie algebra* $\mathfrak{g}$*. Then there exists a unique multiplicative function* $\sigma : \mathbb{G} \to \mathbb{R}$ *and a unique* σ*-multiplicative bivector* Λ *on* $\mathbb{G}$ *satisfying* $(d\sigma)(\mathfrak{e}) = \phi_0$ *and the intrinsic derivative of bivector at* $\mathfrak{e}$ *is* $-d_{*X_0}$*, that is,* $d_{\mathfrak{e}}\Lambda = -d_{*X_0}$*. Furthermore, the following identity holds:*

$$\Lambda^{\#}(d\sigma) = \tilde{X}_0 - e^{-\sigma}\bar{X}_0,$$

and the pair (Λ, E) *is a Jacobi structure on* $\mathbb{G}$*.*

Definition 2.15. A coboundary Jacobi–Lie bialgebra is a Jacobi–Lie bialgebra such that d_{*X_0} is a 1-coboundary, that is, there exists $r \in \wedge^2\mathfrak{g}$ satisfying

$$d_{*X_0}X = ad_{(\phi_0,1)}(X)(r), \quad \forall X \in \mathfrak{g}, \tag{2.5}$$

(for more details, see [11]).

A Jacobi structure on $\mathbb{G}$ was determined in [11, 20] as follows:

$$\Lambda = \tilde{r} - e^{-\sigma}\bar{r}, \quad E = -\tilde{X}_0, \tag{2.6}$$

where both $\tilde{r}$ and $\bar{r}$ $\big(\forall g \in \mathbb{G}\ \tilde{r}(g) = (R_g)_* r$ and $\bar{r}(g) = (L_g)_* r\big)$ are the right and the left invariant bivectors on the Lie group $\mathbb{G}$ (i.e., the pullback of the r-matrix $r \in \wedge^2\mathfrak{g}$) as well as $\tilde{X}_0$ is a right invariant vector field on $\mathbb{G}$. Furthermore, $(d\sigma)(\mathfrak{e}) = \phi_0$.

The relation (2.6) can be expressed in terms of the local coordinates x^μ on $\mathbb{G}$ as follows [20]:

$$\Lambda = \frac{1}{2} r^{ij} (X_i^{R\mu} X_j^{R\nu} - e^{-\sigma} X_i^{L\mu} X_j^{L\nu}) \partial_\mu \wedge \partial_\nu, \tag{2.7}$$

and

$$E = -\alpha^i X_i^{R\mu} \partial_\mu, \tag{2.8}$$

where both $X_i^{R\mu}$ and $X_i^{L\nu}$ are the ith components of the right and the left invariant vector fields on the Lie group $\mathbb{G}$ [20]; namely $X_i^R = X_i^{R\mu}\partial_\mu$ and $X_i^L = X_i^{L\mu}\partial_\mu$. Furthermore, we have that

$$[X_i^R, X_j^R] = f_{ij}^k X_k^R, \qquad [X_i^L, X_j^L] = -f_{ij}^k X_k^L,$$

where f_{ij}^k are the structure constants of the Lie algebra $\mathfrak{g}$ of the Lie group $\mathbb{G}$. Moreover, r^{ij} is the component of the skew-symmetric tensor $r = \frac{1}{2} r^{ij} X_i \wedge Xj$, and the multiplicative function $\sigma : \mathbb{G} \to \mathbb{R}$ is defined as $(d\sigma)(\mathfrak{e}) = \phi_0$ as well as α^i is obtained from the relation $X_0 = \alpha^i X_i$, where $\{X_i\}$ is the basis of the Lie algebra $\mathfrak{g}$. Now, using the results from [20] and the relations (2.7) and (2.8), one can calculate the vector field E and the bivector Λ related to the real two- and three-dimensional Jacobi–Lie bialgebras [20]. Our results are listed in Tables 5.1–5.3. The first column gives the names of the real two- and three-dimensional Jacobi–Lie bialgebras according to [20], and the second column gives the Jacobi structure (Λ, E) on the Lie group $\mathbb{G}$ related to the Lie algebra $\mathfrak{g}$. Note that the elements of the real two-dimensional Jacobi–Lie group $\mathbb{G}$ are given by $g = e^{xX_1}e^{yX_2}$ for all $g \in \mathbb{G}$. Additionally, for the real three-dimensional Jacobi–Lie group $\mathbb{G}$, we assume that $g = e^{xX_1}e^{yX_2}e^{zX_3}$ for all $g \in \mathbb{G}$.

Definition 2.16 ([20])**.** A bi-r-matrix Jacobi–Lie bialgebra is a Jacobi–Lie bialgebra possessing classical r-matrices $r \in \wedge^2\mathfrak{g}$ and $\tilde{r} \in \wedge^2\mathfrak{g}^*$.

Definition 2.17 ([20])**.** A coboundary Jacobi–Lie bialgebra is a Jacobi–Lie bialgebra admitting the classical r-matrix $r \in \wedge^2\mathfrak{g}$ or $\tilde{r} \in \wedge^2\mathfrak{g}^*$.

In [20], real two- and three-dimensional bi-r-matrix Jacobi–Lie bialgebras and coboundary Jacobi–Lie bialgebras have been classified, and here we apply those classifications to obtain the results of Tables 5.1–5.3.

2.3. Jacobi–Lie Hamiltonian systems. Jacobi–Lie Hamiltonian systems were introduced in [10]. In other words, Lie systems possessing a Vessiot–Guldberg Lie algebra of Hamiltonian functions with respect to a Jacobi structure. One of the most useful constructions on Jacobi manifolds is in a sense analogue of the gradient defined as follows.

Definition 2.18. A vector field X on the manifold M is Hamiltonian relative to a Jacobi structure (M, Λ, E) if there exists a function f, referred to as Hamiltonian, such that

$$X_f = [\Lambda, f] + fE = \Lambda^{\#}(df) + fE.$$

Let f be a smooth function on the Jacobi manifold M. Then there exists a unique vector field X_f on M, referred to as a Hamiltonian vector field associated with f, such that the following equality is satisfied:

$$\{f, g\}_{\Lambda,E} = X_f g - g\,Ef, \quad g \in C^\infty(M).$$

A vector field X_f can possess several Hamiltonian functions. Obviously, $(\mathrm{Ham}(M, \Lambda, E), [\cdot, \cdot)$ is a Lie algebra, where the bracket is the one concerning vector fields and $\mathrm{Ham}(M, \Lambda, E)$ is the space of Hamiltonian vector fields of the Jacobi manifold. If $Ef = 0$ (that is, the derivative of the function f in the direction of the vector field E is equal to zero), then the function f is called a *good Hamiltonian function* and X_f, a *good Hamiltonian vector field* [10].

Definition 2.19 ([10])**.** A Jacobi–Lie system is a quadruple (M, Λ, E, X), where (M, Λ, E) is a Jacobi manifold and X is a Lie system such that $\mathfrak{g}^X \subset \mathrm{Ham}(M, \Lambda, E)$.

Definition 2.20 ([10])**.** A Jacobi–Lie Hamiltonian system is a quadruple (M, Λ, E, f), where (M, Λ, E) is a Jacobi manifold and $f : \mathbb{R}\times M \to M, \quad (t, x) \mapsto f_t(x)$ is a t-dependent function and $\mathrm{Lie}(\{f_t\}_{t\in\mathbb{R}}, \{\cdot, \cdot\}_{\Lambda,E})$ is finite dimensional. The system X on M is said to be Jacobi–Lie Hamiltonian system related to (M, Λ, E, f) if X_t is a Hamiltonian vector field with the Hamiltonian function f_t (relative to the Jacobi manifold) for all $t \in \mathbb{R}$.

Theorem 2.21 ([10])**.** *If (M, Λ, E, f) is a Jacobi–Lie Hamiltonian system, then the system X of the form $X_t = X_{f_t}, \forall t \in \mathbb{R}$, is a Jacobi–Lie system (M, Λ, E, X). If X is a Lie system whose $\{X_t\}_{t\in\mathbb{R}}$ are good Hamiltonian vector fields, then X possesses a Jacobi–Lie Hamiltonian.*

3. Jacobi–Lie Hamiltonian systems on real low-dimensional Jacobi–Lie groups

Now we consider some Jacobi structures obtained by using the real two- and three-dimensional Jacobi-Lie groups related to the Jacobi–Lie bialgebras (see Tables 5.1, 5.3 below). In these examples, we consider the Lie group $\mathbb{G}$ related to the Jacobi–Lie bialgebra $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$. For these examples, we use the formalisms mentioned in the previous section for calculating Jacobi–Lie Hamiltonian systems on real low-dimensional Jacobi–Lie groups.

Example 3.1. Consider the real two-dimensional bi-r-matrix Jacobi–Lie bialgebra $((\mathfrak{g},\phi_0),(\mathfrak{g}^*,X_0)) = ((A_2,b\tilde{X}^2),(A_2.i,-bX_1))$, where $X_0 = -bX_1 = \alpha^i X_i \Rightarrow \alpha^1 = -b$, $\alpha^2 = 0$.

In view of the results obtained in [20], we see that the Lie group $\mathbb{A}_2$ with Lie algebra A_2 is $X_1^R = \partial_x, X_2^R = -x\partial_x + \partial_y, X_1^L = e^{-y}\partial_x$, and $X_2^L = \partial_y$, where X_i^R and X_i^L are the ith components of the right and the left invariant vector fields on the Lie group $\mathbb{A}_2$. Moreover, $r = \frac{1}{2}r^{ij}X_i \wedge Xj = X_1 \wedge X_2 \Rightarrow r^{12} = 2$ and $\sigma = by$ [20]. Meanwhile, consider the Lie group $\mathbb{A}_2$ with the coordinates x and y related to the Lie algebra A_2. Hence, using the relations (2.7), (2.8), one can show that the Jacobi bivector field and the Reeb vector field have the following forms (see Table 5.1):

$$\Lambda_{\mathbb{A}_2} = (1 - e^{-(b+1)y})\partial_x \wedge \partial_y, \quad E_{\mathbb{A}_2} = b\partial_x. \tag{3.1}$$

Simple calculations show that they satisfy

$$[\Lambda_{\mathbb{A}_2}, \Lambda_{\mathbb{A}_2}] = 2E_{\mathbb{A}_2} \wedge \Lambda_{\mathbb{A}_2} = 0, \quad [E_{\mathbb{A}_2}, \Lambda_{\mathbb{A}_2}] = 0.$$

Thus, $(\mathbb{A}_2, \Lambda_{\mathbb{A}_2}, E_{\mathbb{A}_2})$ is a Jacobi manifold. Now, using the above results and the bundle morphism

$$\begin{aligned} &\Lambda^{\#} : T^*M \to TM \\ &dg(\Lambda^{\#}(df)) = \Lambda(df, dg), \quad \forall df, dg \in T^*M, \end{aligned} \tag{3.2}$$

we see that

$$\Lambda^{\#}_{\mathbb{A}_2}(df) = (1 - e^{-(b+1)y})\left(-\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right). \tag{3.3}$$

This morphism allows us to relate for every function $f \in C^\infty(M)$ an associated Hamiltonian vector field X_f through the relation

$$X_f = \Lambda^{\#}(df) + fE. \tag{3.4}$$

Substituting (3.3) and $E_{\mathbb{A}_2} = b\partial_x$ in (3.4), we obtain

$$X_f = \left(-(1 - e^{-(b+1)y})\frac{\partial f}{\partial y} + bf\right)\partial_x + \left(1 - e^{-(b+1)y}\right)\frac{\partial f}{\partial x}\partial_y. \tag{3.5}$$

Since every Hamiltonian function f_i induces a unique Hamiltonian vector field, it makes sense to represent it by X_i^H. Thus, one has that

$$\begin{aligned} X_{f_1} = X_1^H &= \left(-\left(1 - e^{-(b+1)y}\right)\frac{\partial f_1}{\partial y} + bf_1\right)\partial_x + \left(1 - e^{-(b+1)y}\right)\frac{\partial f_1}{\partial x}\partial_y, \\ X_{f_2} = X_2^H &= \left(-\left(1 - e^{-(b+1)y}\right)\frac{\partial f_2}{\partial y} + bf_2\right)\partial_x + \left(1 - e^{-(b+1)y}\right)\frac{\partial f_2}{\partial x}\partial y. \end{aligned}$$

Here, the Hamiltonian vector fields X_1^H, X_2^H are linearly independent over $\mathbb{A}_2$ and they form a basis for the Lie algebra A_2 with non-zero commutators $[X_1^H, X_2^H] =$

X_1^H. Note that the Hamiltonian vector fields need not be linearly independent on a Lie group. It is easy to check that

$$X_1^H = b\, e^{\frac{-x}{b}}\, \partial_x + \frac{(-1+e^{-(1+b)y})e^{\frac{-x}{b}}}{b}\, \partial_y, \quad X_2^H = b\partial_x,$$

span the Lie algebra A_2 of Hamiltonian vector fields on $\mathbb{A}_2$, i.e., $[X_1^H, X_2^H] = X_1^H$.

Consider now the system on $\mathbb{A}_2$ defined by

$$\frac{d\alpha_2}{dt} = \sum_{i=1}^{2} b_i(t) X_i^H(\alpha_2), \quad \alpha_2 \in \mathbb{A}_2, \tag{3.6}$$

for arbitrary t-dependent functions $b_i(t)$. $X^{\mathbb{A}_2}$ is a Lie system since the associated t-dependent vector field $X^{\mathbb{A}_2} = \sum_{i=1}^{2} b_i(t) X_i^H$ takes values in the Lie algebra $\langle X_1^H, X_2^H\rangle$, where $[X_1^H, X_2^H] = X_1^H$, namely the Lie algebra A_2.

We now illustrate that the Lie system (3.6) is a Jacobi–Lie system. Indeed, X_1^H and X_2^H are Hamiltonian vector fields with respect to $(\mathbb{A}_2, \Lambda_{\mathbb{A}_2}, E_{\mathbb{A}_2})$ with the Hamiltonian functions $f_1 = e^{\frac{-x}{b}}$ and $f_2 = 1$, respectively (i.e., $X_i^H = \Lambda^{\#}(df_i) + f_i E$), and thus $(\mathbb{A}_2, \Lambda_{\mathbb{A}_2}, E_{\mathbb{A}_2}, X^{\mathbb{A}_2})$ is a Jacobi–Lie system.

Since $f = \sum_{i=1}^{2} b_i(t) f_i = b_1(t) e^{\frac{-x}{b}} + b_2(t)$ is a Hamiltonian function of $X^{\mathbb{A}_2}$ for every $t \in \mathbb{R}$ and the functions f_1 and f_2 satisfy the commutation relation $\{f_1, f_2\}_{\Lambda_{\mathbb{A}_2}, E_{\mathbb{A}_2}} = f_1$, then the functions $\{f_t\}_{t\in\mathbb{R}}$ span a finite-dimensional real Lie algebra of functions with respect to the Lie bracket induced by (3.1). Consequently, $X^{\mathbb{A}_2}$ admits a Jacobi–Lie Hamiltonian system $(\mathbb{A}_2, \Lambda_{\mathbb{A}_2}, E_{\mathbb{A}_2}, f)$.

Example 3.2. Consider the real three-dimensional bi-r-matrix Jacobi–Lie bialgebra $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0)) = ((III, -b\tilde{X}^2 + b\tilde{X}^3), (III.iv, bX_1))$, where $X_0 = bX_1 = \alpha^i X_i \Rightarrow \alpha^1 = b$, $\alpha^2 = 0, \alpha^3 = 0$. In view of the results of [20], we see that the Lie group $\mathbb{III}$ with Lie algebra III is

$$\begin{pmatrix} X_1^R \\ X_2^R \\ X_3^R \end{pmatrix} = \begin{pmatrix} \partial_x \\ \frac{e^{2x}+1}{2}\partial_y + \frac{e^{2x}-1}{2}\partial_z \\ \frac{e^{2x}-1}{2}\partial_y + \frac{e^{2x}+1}{2}\partial_z \end{pmatrix}, \quad \begin{pmatrix} X_1^L \\ X_2^L \\ X_3^L \end{pmatrix} = \begin{pmatrix} \partial_x + (y+z)(\partial_y + \partial_z) \\ \partial_y \\ \partial_z \end{pmatrix}, \tag{3.7}$$

where X_i^R and X_i^L are the ith components of the right and the left invariant vector fields on the Lie group $\mathbb{III}$. Moreover, $r = \frac{1}{2} r^{ij} X_i \wedge Xj = \frac{1}{2} X_1 \wedge X_2 - \frac{1}{2} X_1 \wedge X_3 \Rightarrow r^{12} = 1$, $r^{13} = -1$ and $\sigma = -b(y-z)$ [20]. Meanwhile, consider the Lie group $\mathbb{III}$ with the coordinates x, y and z related to Lie algebra III. Hence, using the relations (2.7), (2.8), one can show that the Jacobi bivector field and the Reeb vector field have the following forms (see Table 2):

$$\begin{aligned} \Lambda_{\mathbb{III}} &= \frac{1}{2}(1 - e^{b(y-z)})\partial_x \wedge \partial_y - \frac{1}{2}(1 - e^{b(y-z)})\partial_x \wedge \partial_z + (y+z)e^{b(y-z)}\partial_y \wedge \partial_z, \\ E_{\mathbb{III}} &= -b\partial_x. \end{aligned} \tag{3.8}$$

Obviously, $[\Lambda_{\mathbb{III}}, \Lambda_{\mathbb{III}}] = 2E_{\mathbb{III}} \wedge \Lambda_{\mathbb{III}} = -2b(y+z)e^{b(y-z)}\partial_x \wedge \partial_y \wedge \partial_z$ and $[E_{\mathbb{III}}, \Lambda_{\mathbb{III}}] = 0$. As a result, $(\mathbb{III}, \Lambda_{\mathbb{III}}, E_{\mathbb{III}})$ is a Jacobi manifold. Now, using the above results and (3.2), it follows that

$$\begin{aligned}\Lambda^{\#}_{\mathbb{III}}(df) = &\left(-\frac{\left(1-e^{b(y-z)}\right)}{2}\frac{\partial f}{\partial y} + \frac{\left(1-e^{b(y-z)}\right)}{2}\frac{\partial f}{\partial z}\right)\partial_x \\ &+ \left(\frac{\left(1-e^{b(y-z)}\right)}{2}\frac{\partial f}{\partial x} - (y+z)\, e^{b(y-z)}\frac{\partial f}{\partial z}\right)\partial_y \\ &+ \left(\frac{\left(1-e^{b(y-z)}\right)}{2}\frac{\partial f}{\partial x} - (y+z)\, e^{b(y-z)}\frac{\partial f}{\partial_z}\right)\partial z \qquad (3.9)\end{aligned}$$

Substituting (3.9) and $E_{\mathbb{III}} = -b\partial_x$ in (3.4), we obtain

$$\begin{aligned}X_f = &\left(-\frac{\left(1-e^{b(y-z)}\right)}{2}\frac{\partial f}{\partial y} + \frac{\left(1-e^{b(y-z)}\right)}{2}\frac{\partial f}{\partial z} - bf\right)\partial_x \\ &+ \left(\frac{\left(1-e^{b(y-z)}\right)}{2}\frac{\partial f}{\partial x} - (y+z)\, e^{b(y-z)}\frac{\partial f}{\partial z}\right)\partial_y \\ &+ \left(\frac{\left(1-e^{b(y-z)}\right)}{2}\frac{\partial f}{\partial x} - (y+z)\, e^{b(y-z)}\frac{\partial f}{\partial z}\right)\partial_z\end{aligned}$$

Since every Hamiltonian function f_i induces a unique Hamiltonian vector field, it makes sense to represent it by X_i^H. Thus, one has that $X_1^H = X_{f_1}$, $X_2^H = X_{f_1}$ and $X_3^H = X_{f_3}$. The Hamiltonian vector fields X_1^H, X_2^H and X_3^H are linearly independent over $\mathbb{III}$ and they form a basis for the Lie algebra II with non-zero commutators $[X_2^H, X_3^H] = X_1^H$.

It is straightforward to verify that the Lie algebra II of Hamiltonian vector fields on $\mathbb{III}$ is spanned by

$$\begin{aligned}X_1^H &= -b\,\partial_x, \\ X_2^H &= \left(-\frac{1}{2} + \frac{1}{2}e^{b(y-z)} - by\right)\partial_x + (y+z)e^{b(y-z)}\,\partial_z, \\ X_3^H &= Ei(1, -b(y+z))e^{-b(y+z)}b\partial_x - \partial_y \\ &\quad + \left(2\,b\,(y+z)\,Ei\,(1, -b\,(y+z))\,e^{-b(y+z)} + 1\right)\partial_z,\end{aligned}$$

where $Ei\,(1, -b\,(y+z)) = \int_{-b(y+z)}^{\infty} \frac{e^{-x}}{x} dx$. The system on $\mathbb{III}$ can be defined as

$$\frac{d\gamma}{dt} = \sum_{i=1}^{3} b_i(t) X_i^H(\gamma), \qquad \gamma \in \mathbb{III}, \qquad (3.10)$$

for arbitrary t-dependent functions $b_i(t)$.

As the associated t-dependent vector field $X^{\mathbb{III}} = \sum_{i=1}^{3} b_i(t) X_i^H$ takes values in the Lie algebra II, that is, $[X_2^H, X_3^H] = X_1^H$, then $X^{\mathbb{III}}$ is a Lie system. We now manifest that $(\mathbb{III}, \Lambda_{\mathbb{III}}, E_{\mathbb{III}}, X^{\mathbb{III}})$ is a Jacobi–Lie system. In fact, X_1^H, X_2^H

and X_3^H are Hamiltonian vector fields relative to $(\mathbb{III}, \Lambda_{\mathbb{III}}, E_{\mathbb{III}})$ with good Hamiltonian functions $f_1 = 1$, $f_2 = y$, and $f_3 = -e^{-2by}Ei(1, -b(y+z)$, respectively, $\big($i.e., $X_i^H = \Lambda^{\#}(df_i) + f_i E\big)$, and thus $(\mathbb{III}, \Lambda_{\mathbb{III}}, E_{\mathbb{III}}, X^{\mathbb{III}})$ is a Jacobi–Lie system. Because $f = \sum_{i=1}^{2} b_i(t) f_i = b_1(t) + b_2(t)y - b_3(t)e^{-2by}Ei(1, -b(y+z)$ is a Hamiltonian function of $X^{\mathbb{III}}$ for every $t \in \mathbb{R}$, and the functions f_1, f_2, and f_3 satisfy the commutation relations $\{f_2, f_3\}_{\Lambda_{\mathbb{III}}, E_{\mathbb{III}}} = f_1$, then the functions $\{f_t\}_{t\in\mathbb{R}}$ span a finite-dimensional real Lie algebra of functions with respect to the Lie bracket induced by (3.8). Consequently, $X^{\mathbb{III}}$ admits a Jacobi–Lie Hamiltonian system $(\mathbb{III}, \Lambda_{\mathbb{III}}, E_{\mathbb{III}}, f)$.

Example 3.3. Consider the real three-dimensional bi-r-matrix Jacobi–Lie bialgebra $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0)) = ((III, 0), (III.v, \frac{1}{2}X_2 - \frac{1}{2}X_3))$, where $X_0 = \frac{1}{2}X_2 - \frac{1}{2}X_3 = \alpha^i X_i \Rightarrow \alpha^1 = 0$, $\alpha^2 = \frac{1}{2}$ and $\alpha^3 = -\frac{1}{2}$.

In view of the results of [20], we see that the Lie group $\mathbb{III}$ with Lie algebra III is

$$\begin{pmatrix} X_1^R \\ X_2^R \\ X_3^R \end{pmatrix} = \begin{pmatrix} \partial_x \\ \frac{e^{2x}+1}{2}\partial_y + \frac{e^{2x}-1}{2}\partial_z \\ \frac{e^{2x}-1}{2}\partial_y + \frac{e^{2x}+1}{2}\partial_z \end{pmatrix}, \quad \begin{pmatrix} X_1^L \\ X_2^L \\ X_3^L \end{pmatrix} = \begin{pmatrix} \partial_x + (y+z)(\partial_y + \partial_z) \\ \partial_y \\ \partial_z \end{pmatrix}, \tag{3.11}$$

where X_i^R and X_i^L are the ith components of the right and the left invariant vector fields on the Lie group $\mathbb{III}$. Moreover, $r = \frac{1}{2}r^{ij}X_i \wedge Xj = \frac{1}{2}X_1 \wedge X_3 + \frac{1}{2}X_2 \wedge X_3 \Rightarrow r^{13} = 1$, $r^{23} = 1$ and $\sigma = 0$ [20]. Meanwhile, consider the Lie group $\mathbb{III}$ with the coordinates x, y and z related to the Lie algebra III. Hence, using the relations (2.7), (2.8), one can show that the Jacobi bivector field and the Reeb vector field have the following forms (see Table 5.2):

$$\begin{aligned} \Lambda_{\mathbb{III}} &= \frac{e^{2x}-1}{4}\partial_x \wedge \partial_y + \frac{e^{2x}-1}{4}\partial_x \wedge \partial_z - \frac{1}{2}(y+z+1-e^{2x})\partial_y \wedge \partial_z, \\ E_{\mathbb{III}} &= -\frac{1}{2}\partial_y + \frac{1}{2}\partial_z. \end{aligned} \tag{3.12}$$

One can show that

$$[\Lambda_{\mathbb{III}}, \Lambda_{\mathbb{III}}] = 2E_{\mathbb{III}} \wedge \Lambda_{\mathbb{III}} = \frac{e^{2x}-1}{2}\partial_x \wedge \partial_y \wedge \partial_z, \quad [E_{\mathbb{III}}, \Lambda_{\mathbb{III}}] = 0.$$

Hence, $(\mathbb{III}, \Lambda_{\mathbb{III}}, E_{\mathbb{III}})$ is a Jacobi manifold. Now, using the above results and (3.2), it follows that

$$\begin{aligned} \Lambda^{\#}_{\mathbb{III}}(df) = &\left(\frac{(1-e^{2x})}{4}\frac{\partial f}{\partial y} + \frac{(1-e^{2x})}{4}\frac{\partial f}{\partial z}\right)\partial_x \\ &+ \left(\frac{(-1+e^{2x})}{4}\frac{\partial f}{\partial x} + \frac{(y+z+1-e^{2x})}{2}\frac{\partial f}{\partial z}\right)\partial_y \\ &+ \left(\frac{(-1+e^{2x})}{4}\frac{\partial f}{\partial x} - \frac{(y+z+1-e^{2x})}{2}\frac{\partial f}{\partial y}\right)\partial_z. \end{aligned} \tag{3.13}$$

Substituting (3.13) and $E_{\mathbb{III}} = -\frac{1}{2}\partial_y + \frac{1}{2}\partial_z$ in (3.4), we obtain

$$X_f = \left(\frac{\left(1-e^{2\,x}\right)}{4}\frac{\partial f}{\partial y} + \frac{\left(1-e^{2\,x}\right)}{4}\frac{\partial f}{\partial z} \right)\partial_x + \left(\frac{\left(-1+e^{2\,x}\right)}{4}\frac{\partial f}{\partial x} + \frac{\left(y+z+1-e^{2\,x}\right)}{2}\frac{\partial f}{\partial z} - \frac{f}{2} \right)\partial_y + \left(\frac{\left(-1+e^{2\,x}\right)}{4}\frac{\partial f}{\partial x} - \frac{\left(y+z+1-e^{2\,x}\right)}{2}\frac{\partial f}{\partial y} + \frac{f}{2} \right)\partial_z.$$

Since every Hamiltonian function f_i induces a unique Hamiltonian vector field, it makes sense to represent it by X_i^H. Thus, one has that $X_1^H = X_{f_1}$, $X_2^H = X_{f_1}$, and $X_3^H = X_{f_3}$. The Hamiltonian vector fields X_1^H, X_2^H, and X_3^H are linearly independent over $\mathbb{III}$ and they form a basis for the Lie algebra III with non-zero commutators $[X_1^H, X_2^H] = -(X_2^H + X_3^H)$, $[X_1^H, X_3^H] = -(X_2^H + X_3^H)$.

A simple calculation shows that

$$\begin{aligned} X_1^H &= -\frac{1}{2}\left(2\,x-3\,y-z-2\right)\left(-1+e^{2\,x}\right)\partial_x \\ &\quad + \left(\frac{3}{2}\,e^{2\,x}y + \frac{1}{2}\,e^{2\,x}z + e^{2\,x} + \frac{y}{2} - \frac{z}{2} - 1 - e^{2\,x}x - x\right)\partial_y \\ &\quad + \Big(-\frac{3}{2}\,e^{2\,x}y - \frac{1}{2}\,e^{2\,x}z - e^{2\,x} + \frac{3}{2}\,y + \frac{5}{2}\,z \\ &\qquad + 1 + e^{2\,x}x + y^2 + 2\,yz + z^2 + x\Big)\partial_z, \\ X_2^H &= \left(\frac{1}{2} - \frac{1}{2}e^{2x}\right)\partial_x - \frac{1}{2}e^{2x}\,\partial_y + \frac{1}{2}\,e^{2x}\,\partial_z, \\ X_3^H &= -\frac{1}{2}\partial_y + \frac{1}{2}\partial_z, \end{aligned}$$

span the Lie algebra III of Hamiltonian vector fields on $\mathbb{III}$.

The system on $\mathbb{III}$ can be written as

$$\frac{d\gamma}{dt} = \sum_{i=1}^{3} b_i(t) X_i^H(\gamma), \qquad \gamma \in \mathbb{III},$$

for arbitrary t-dependent functions $b_i(t)$. Since the associated t-dependent vector field $X^{\mathbb{III}} = \sum_{i=1}^{3} b_i(t) X_i^H$ takes values in the Lie algebra III, that is, $[X_1^H, X_2^H] = -(X_2^H + X_3^H), [X_1^H, X_3^H] = -(X_2^H + X_3^H)$, then $X^{\mathbb{III}}$ is a Lie system.

We now prove that $(\mathbb{III}, \Lambda_{\mathbb{III}}, E_{\mathbb{III}}, X^{\mathbb{III}})$ is a Jacobi–Lie system. As a matter of the fact, X_1^H, X_2^H and X_3^H are Hamiltonian vector fields relative to $(\mathbb{III}, \Lambda_{\mathbb{III}}, E_{\mathbb{III}})$ with the Hamiltonian functions $f_1 = 2\,(y+z+2)\,(-y+x)$, $f_2 = 1+y+z$ and $f_3 = 1$, respectively, (i.e., $X_i^H = \Lambda^{\#}(df_i) + f_i E$), and thus $(\mathbb{III}, \Lambda_{\mathbb{III}}, E_{\mathbb{III}},, X^{\mathbb{III}})$ is a Jacobi–Lie system.

Using the Lie bracket induced by $\Lambda_{\mathbb{III}}$ and $E_{\mathbb{III}}$ of the Lie group $\mathbb{III}$, we can write $\{f_1, f_2\}_{\Lambda_{\mathbb{III}}, E_{\mathbb{III}}} = -f_2 - f_3$, $\{f_1, f_3\}_{\Lambda_{\mathbb{III}}, E_{\mathbb{III}}} = -f_2 - f_3$. Therefore, $(\mathbb{III}, \Lambda_{\mathbb{III}}, E_{\mathbb{III}}, f = \sum_{i=1}^{3} b_i(t) f_i)$ for $X^{\mathbb{III}}$ is a Jacobi–Lie Hamiltonian system, where $\Lambda_{\mathbb{III}}$ and $E_{\mathbb{III}}$ are those appearing in (3.12).

Example 3.4. Consider the real three-dimensional bi-r-matrix Jacobi–Lie bialgebra $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0)) = ((IV, -\tilde{X}^1), (III.vi, -X_2 - X_3))$, where $X_0 = -X_2 - X_3 = \alpha^i X_i \Rightarrow \alpha^1 = 0$, $\alpha^2 = -1$ and $\alpha^3 = -1$.

In view of the results of [20], we see that the Lie group $\mathbb{IV}$ with Lie algebra IV is

$$\begin{pmatrix} X_1^R \\ X_2^R \\ X_3^R \end{pmatrix} = \begin{pmatrix} \partial_x \\ e^x \partial_y - x e^x \partial_z \\ e^x \partial_z \end{pmatrix}, \quad \begin{pmatrix} X_1^L \\ X_2^L \\ X_3^L \end{pmatrix} = \begin{pmatrix} \partial_x + y\partial_y - (y - z)\partial_z \\ \partial_y \\ \partial_z \end{pmatrix},$$

where X_i^R and X_i^L are the ith components of the right and the left invariant vector fields on the Lie group $\mathbb{IV}$. Moreover, $r = \frac{1}{2} r^{ij} X_i \wedge Xj = X_1 \wedge X_2 + X_2 \wedge X_3 \Rightarrow r^{12} = 2$, $r^{23} = 2$ and $\sigma = -x$ [20]. Meanwhile, consider the Lie group $\mathbb{IV}$ with the coordinates x, y and z related to the Lie algebra IV. Hence, using the relations (2.7), (2.8), one can show that the Jacobi bivector field and the Reeb vector field have the following forms (see Table 5.2):

$$\Lambda_{\mathbb{IV}} = -x e^x \partial_x \wedge \partial_z + e^x (z - y - 1 + e^x) \partial_y \wedge \partial_z, \quad E_{\mathbb{IV}} = e^x \partial_y + e^x (1 - x) \partial_z, \tag{3.14}$$

Then one can show that

$$[\Lambda_{\mathbb{IV}}, \Lambda_{\mathbb{IV}}] = 2\, x e^{2\,x} \partial_x \wedge \partial_y \wedge \partial_z = 2 E_{\mathbb{IV}} \wedge \Lambda_{\mathbb{IV}}, \quad [E_{\mathbb{IV}}, \Lambda_{\mathbb{IV}}] = 0.$$

Hence $(\mathbb{IV}, \Lambda_{\mathbb{IV}}, E_{\mathbb{IV}})$ is a Jacobi manifold. Now, using the above results and (3.2), it follows that

$$\begin{aligned} \Lambda^{\#}_{\mathbb{IV}}(df) = {} & \left(x e^x \frac{\partial f}{\partial z} \right) \partial_x + \left(-e^x \left(z - y - 1 + e^x\right) \frac{\partial f}{\partial z} \right) \partial_y \\ & + \left(-x e^x \frac{\partial f}{\partial x} + e^x \left(z - y - 1 + e^x\right) \frac{\partial f}{\partial y} \right) \partial_z. \end{aligned} \tag{3.15}$$

Substituting (3.15) and $E_{\mathbb{IV}} = e^x \partial_y + e^x (1 - x) \partial_z$ in (3.4), we obtain

$$\begin{aligned} X_f = {} & \left(x e^x \frac{\partial f}{\partial z} \right) \partial_x + \left(-e^x \left(z - y - 1 + e^x\right) \frac{\partial f}{\partial z} + f e^x \Big(\right) \partial_y \\ & + \left(-x e^x \frac{\partial f}{\partial x} + e^x \left(z - y - 1 + e^x\right) \frac{\partial f}{\partial y} + e^x \left(1 - x\right) f \right) \partial_z. \end{aligned}$$

Since every Hamiltonian function f_i induces a unique Hamiltonian vector field, it makes sense to represent it by X_i^H. Thus, one has that $X_1^H = X_{f_1}$, $X_2^H = X_{f_1}$, and $X_3^H = X_{f_3}$. The Hamiltonian vector fields X_1^H, X_2^H, and X_3^H are linearly independent over $\mathbb{IV}$ and they form a basis for the Lie algebra IV with non-zero commutators $[X_1^H, X_2^H] = -(X_2^H - X_3^H)$, $[X_1^H, X_3^H] = -X_3^H$. A short calculation shows that

$$\begin{aligned}
X_1^H &= -\partial_x + \frac{y - 1 + e^x}{x}\,\partial_y + \frac{2\,e^x + 2\,y - 2}{x}\,\partial_z,\\
X_2^H &= e^x\,\partial_y + e^x(1-x)\,\partial_z,\\
X_3^H &= -\frac{e^x}{x}\,\partial_y + \frac{e^x\,(x-2)}{x}\,\partial_z,
\end{aligned}$$

span the Lie algebra IV of Hamiltonian vector fields on $\mathbb{IV}$. The system on $\mathbb{IV}$ can be considered as

$$\frac{d\delta}{dt} = \sum_{i=1}^{3} b_i(t) X_i^H(\delta), \qquad \forall \delta \in \mathbb{IV},$$

for arbitrary t-dependent functions $b_i(t)$. Since the associated t-dependent vector field $X^{\mathbb{IV}} = \sum_{i=1}^{3} b_i(t) X_i^H$ takes values in the Lie algebra IV, that is, $[X_1^H, X_2^H] = -(X_2^H - X_3^H)$, $[X_1^H, X_3^H] = -X_3^H$, then $X^{\mathbb{IV}}$ is a Lie system. We now exhibit that $(\mathbb{IV}, \Lambda_{\mathbb{IV}}, E_{\mathbb{IV}}, X^{\mathbb{IV}})$ is a Jacobi–Lie system. Actually, X_1^H, X_2^H, and X_3^H are the Hamiltonian vector fields relative to $(\mathbb{IV}, \Lambda_{\mathbb{IV}}, E_{\mathbb{IV}})$ with the Hamiltonian functions $f_1 = \frac{(2\,y - z)e^{-x}}{x}$, $f_2 = 1$, and $f_3 = -\frac{1}{x}$ respectively (i.e., $X_i^H = \Lambda^{\#}(df_i) + f_i E$), and thus $(\mathbb{IV}, \Lambda_{\mathbb{IV}}, E_{\mathbb{IV}}, X^{\mathbb{IV}})$ is a Jacobi–Lie system.

Using the Lie bracket induced by $\Lambda_{\mathbb{IV}}$ and $E_{\mathbb{IV}}$ of the Lie group $\mathbb{IV}$, the functions f_1, f_2, and f_3 satisfy the commutation relations $\{f_1, f_2\}_{\Lambda_{\mathbb{IV}},\, E_{\mathbb{IV}}} = -f_2 + f_3$, $\{f_1, f_3\}_{\Lambda_{\mathbb{IV}},\, E_{\mathbb{IV}}} = -f_3$. Therefore, $(\mathbb{IV}, \Lambda_{\mathbb{IV}}, E_{\mathbb{III}}, f = \sum_{i=1}^{3} b_i(t) f_i)$ for $X^{\mathbb{IV}}$ is a Jacobi–Lie Hamiltonian system, where $\Lambda_{\mathbb{IV}}$ and $E_{\mathbb{IV}}$ are those appearing in (3.14).

4. Lie symmetry for Jacobi–Lie Hamiltonian systems

We now give an example of Jacobi–Lie Hamiltonian systems with good Hamiltonian functions on the Lie group $\mathbb{VIII} = SL(2, \mathbb{R})$ whose distribution associated with this system is of dimension two. Then we obtain a t-independent Lie symmetry [7] for this system to illustrate our procedure.

Let X be a t-dependent vector field on M, the associated distribution of X is the generalized distribution Δ^X on M spanned by the vector fields of $\mathfrak{g}^X$. In other words,

$$\Delta_x^X = \left\{ Z_x \mid Z \in \mathfrak{g}^X \right\} \subset T_x M,$$

and the associated co-distribution of X is the generalized co-distribution $(\Delta^X)^\perp$ on M of the form

$$(\Delta_x^X)^\perp = \left\{ \nu \in T_x^* M \mid \forall Y_x \in \Delta_x^X \ \ \nu(Y_x) = 0 \right\} \subset T_x^* M,$$

where $(\Delta_x^X)^\perp$ is the annihilator of Δ_x^X.

The function $\rho^X : M \to \mathbb{N} \cup \{0\}$, $x \mapsto \dim \Delta_x^X$ is lower semicontinuous at x since it cannot decrease in the neighborhood of x. In addition, $\rho^X(x)$ is constant on the connected components of a dense and open subset U^X of M (cf. [22, p. 19]), where Δ^X becomes a regular involutive distribution. Also, $(\Delta_x^X)^\perp$ becomes a regular co-distribution on each connected component since $\dim(\Delta_x^X)^\perp = \dim M - \rho^X(x)$.

Theorem 4.1. *A function $h : U^X \to \mathbb{R}$ is a local t-independent constant of motion for a t-dependent vector field X if and only if $dh \in (\Delta_x^X)^\perp|_{U^X}$.*

The proof is given in [7].

Definition 4.2. Let X be a Jacobi–Lie system with a Jacobi–Lie Hamiltonian structure (M, Λ, E, f). Then one can define its symmetry distribution as follows: $(S_{\Lambda,E}^X)_x = \Lambda^\#(dh_i) + h_i E \in T_x M$, where $dh_i \in (\Delta_x^X)^\perp|_U$.

Now, using the symmetry distribution, we study the t-independent Lie symmetries of Jacobi–Lie Hamiltonian systems with good Hamiltonian functions on real low-dimensional Lie groups. In the following theorem, we state the result not giving the proof of the theorem [10].

Theorem 4.3. *Let X be a Jacobi–Lie system possessing a Jacobi–Lie Hamiltonian structure (M, Λ, E, f) with good Hamiltonian functions $\{f_t\}_{t\in\mathbb{R}}$. The smooth function h on the Jacobi manifold M is a t-independent constant of motion for X if and only if h commutes with all elements of Lie$(\{f_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_{\Lambda,E})$ relative to $\{\cdot,\cdot\}_{\Lambda,E}$.*

Lemma 4.4. *The mapping*

$$\varphi : (C^\infty(M), \{\cdot,\cdot\}_{\Lambda,E}) \to (\mathrm{Ham}(M, \Lambda, E), [\cdot,\cdot])$$

is a homomorphism Lie algebras, i.e., $\varphi\{f, g\}_{\Lambda,E} = [X_f, X_g]$.

Theorem 4.5. *Let X be a Jacobi–Lie system admitting a Jacobi–Lie Hamiltonian structure (M, Λ, E, f) with good Hamiltonian functions $\{f_t\}_{t\in\mathbb{R}}$. If h is a t-independent constant of motion for X, then $X_h = \Lambda^\#(dh)+hE$ is a t-independent Lie symmetry of X.*

Proof. In view of Lemma 4.4, we have

$$\begin{aligned}[X_h, X_{f_t}] &= [\Lambda^\#(dh) + hE, \Lambda^\#(df_t) + f_t E] = \varphi\{h, f_t\}_{\Lambda,E} \\ &= -\varphi\{f_t, h\}_{\Lambda,E} = -\varphi(X_t h - hE f_t) = -\varphi(X_t h) = 0, \quad t \in \mathbb{R}. \qquad \square\end{aligned}$$

Example 4.6. As an example of the above result, let us consider the Lie group $\mathbb{VIII} = SL(2,\mathbb{R})$ with the following Jacobi structure:

$$\Lambda_{\mathbb{VIII}} = xy\partial_x \wedge \partial_y - (1 + yz)\partial_y \wedge \partial_z, \quad E_{\mathbb{VIII}} = x\partial_x - y\partial_y + z\partial_z, \tag{4.1}$$

where x, y and z are the local coordinates on the Lie group $VIII$ (see [10]).

A simple calculation shows that $X_1^H = -x\partial_z$ and $X_2^H = -y\partial_y + z\partial_z$ span the Lie algebra A_2 of Hamiltonian vector fields on $\mathbb{VIII}$.

The system on $\mathbb{VIII}$ can be written as

$$\frac{d\gamma}{dt} = \sum_{i=1}^{2} b_i(t) X_i^H(\gamma), \quad \gamma \in \mathbb{VIII},$$

for arbitrary t-dependent functions $b_i(t)$. Since the associated t-dependent vector field $X^{\mathbb{VIII}} = \sum_{i=1}^{2} b_i(t) X_i^H$ takes values in the Lie algebra A_2, that is, $[X_1^H, X_2^H] = X_1^H$, then $X^{\mathbb{VIII}}$ is a Lie system.

We now prove that $(\mathbb{VIII}, \Lambda_{\mathbb{VIII}}, E_{\mathbb{VIII}}, X^{\mathbb{VIII}})$ is a Jacobi–Lie system. As a matter of the fact, X_1^H and X_2^H are Hamiltonian vector fields relative to $(\mathbb{VIII}, \Lambda_{\mathbb{VIII}}, E_{\mathbb{VIII}})$ with *good Hamiltonian functions* $f_1 = xy$ and $f_2 = -yz$. Note that these functions are first integrals of X_i^H and $E_{\mathbb{VIII}}$ for $i = 1, 2$, respectively. Subsequently, $(\mathbb{VIII}, \Lambda_{\mathbb{VIII}}, E_{\mathbb{VIII}}, X^{\mathbb{VIII}})$ is a Jacobi–Lie system. Using the Lie bracket induced by $\Lambda_{\mathbb{VIII}}$ and $E_{\mathbb{VIII}}$ of Lie group $\mathbb{VIII}$, we can write $\{f_1, f_2\}_{\Lambda_{\mathbb{VIII}}, E_{\mathbb{VIII}}} = f_1$; therefore, $(\mathbb{VIII}, \Lambda_{\mathbb{VIII}}, E_{\mathbb{VIII}}, f = \sum_{i=1}^{2} b_i(t) f_i)$ for $X^{\mathbb{VIII}}$ is a Jacobi–Lie Hamiltonian system.

It is easy to check that $h = 1$ is a t-independent constant of motion. One can show that

$$\{h\,, f_\alpha\}_{\Lambda_{\mathbb{VIII}}, E_{\mathbb{VIII}}} = 0, \qquad \alpha = 1, 2.$$

Then the function h always Jacobi commutes with the whole Lie algebra $Lie(\{f_t\}_{t\in\mathbb{R}}, \{\cdot,\cdot\}_{\Lambda_{\mathbb{III}}, E_{\mathbb{III}}})$, as expected.

By applying Theorem 4.5, $X_h = \Lambda^{\#}(dh) + hE$ must be a Lie symmetry for this system. A short calculation shows that $X_h = x\partial_x - y\partial_y + z\partial_z$. It is easy to check that X_h commutes with X_1^H, X_2^H, and thus commutes with every X_{f_t}, with $t \in \mathbb{R}$, i.e., X_h is a Lie symmetry for $X^{\mathbb{VIII}}$.

5. Concluding remarks

Using the realizations [18] of the complete list of Jacobi structures on real two- and three-dimensional Jacobi–Lie groups [20], we have obtained Hamiltonian vector fields and achieved Jacobi–Lie Hamiltonian systems on real low-dimensional Jacobi–Lie groups. Then we have presented Lie symmetries for Jacobi–Lie Hamiltonian systems with good Hamiltonian functions.

Table 5.1: Reeb vector field and Jacobi bivector field related to real two-dimensional bi-r-matrix Jacobi–Lie bialgebras.

$((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$	Reeb vector field E and Jacobi bivector field Λ
$((A_1, \tilde{X}^1), (A_1, X_2))$	$E = -\partial_y$ $\Lambda = (1 - e^{-x})\partial_x \wedge \partial_y$
$((A_2, b\tilde{X}^2), (A_2.i, -bX_1))$	$E = b\partial_x$ $\Lambda = (1 - e^{-(b+1)y})\partial_x \wedge \partial_y$
$((A_1, 0), (A_2, -X_2))$	$E = \partial_y$ $\Lambda = 0$

Table 5.2: Reeb vector field and Jacobi bivector field related to real three-dimensional bi-r-matrix Jacobi–Lie bialgebras.

$((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$	Reeb vector field E and Jacobi bivector field Λ
$((I, -\tilde{X}^2 + \tilde{X}^3), (III, -2X_1))$	$E = 2\partial_x$ $\Lambda = (-1 + e^{y-z})\partial_x \wedge \partial_y$ $+(1 - e^{y-z})\partial_x \wedge \partial_z$
$((II, 0), (I, X_1))$	$E = -\partial_x$ $\Lambda = -y\partial_x \wedge \partial_y - z\partial_x \wedge \partial_z$
$((II, 0), (V, bX_1))$	$E = -b\partial_x$ $\Lambda = -(1 + b)y\partial_x \wedge \partial_y$ $-(1 + b)z\partial_x \wedge \partial_z$
$((III, b\tilde{X}^1), (III.i, -X_2 + X_3))$	$E = \partial_y - \partial_z$ $\Lambda = -\frac{1}{b}(1 - e^{-bx})\partial_x \wedge \partial_y$ $+\frac{1}{b}(1 - e^{-bx})\partial_x \wedge \partial_z$ $-\frac{2}{b}(y + z)e^{-bx}\partial_y \wedge \partial_z$
$((III, -b\tilde{X}^2 + b\tilde{X}^3), (III.iv, bX_1))$	$E = -b\partial_x$ $\Lambda = \frac{1}{2}(1 - e^{b(y-z)})\partial_x \wedge \partial_y$ $-\frac{1}{2}(1 - e^{b(y-z)})\partial_x \wedge \partial_z$ $+(y + z)e^{b(y-z)}\partial_y \wedge \partial_z$
$((III, 0), (III.v, \frac{1}{2}X_2 - \frac{1}{2}X_3))$	$E = -\frac{1}{2}\partial_y + \frac{1}{2}\partial_z$ $\Lambda = \frac{e^{2x}-1}{4}\partial_x \wedge \partial_y + \frac{e^{2x}-1}{4}\partial_x \wedge \partial_z$ $-\frac{1}{2}(y + z + 1 - e^{2x})\partial_y \wedge \partial_z$
$((III, 0), (IV.iv, X_2 - X_3))$	$E = -\partial_y + \partial_z$ $\Lambda = \frac{1}{2}(e^{2x} - 1)\partial_y \wedge \partial_z$
$((III, -2\tilde{X}^1), (V.i, -X_2 - X_3))$	$E = e^{2x}\partial_y + e^{2x}\partial_z$ $\Lambda = 0$
$((III, 0), (VI_0.iv, X_2 - X_3))$	$E = -\partial_y + \partial_z$ $\Lambda = -2(y + z)\partial_y \wedge \partial_z$
$((III, 0), (VI_a.vii, -X_2 + X_3))$	$E = \partial_y - \partial_z$ $\Lambda = \frac{-2}{a-1}(y + z)\partial_y \wedge \partial_z$
$((III, 0), (VI_a.viii, -X_2 + X_3))$	$E = \partial_y - \partial_z$ $\Lambda = \frac{2}{a+1}(y + z)\partial_y \wedge \partial_z$
$((IV, -\tilde{X}^1), (III.vi, -X_2 - X_3))$	$E = e^x\partial_y + e^x(1 - x)\partial_z$ $\Lambda = -xe^x\partial_x \wedge \partial_z$ $+e^x(z - y - 1 + e^x)\partial_y \wedge \partial_z$

$((IV, -\tilde{X}^1), (IV.i, -bX_3))$	$E = be^x \partial_z$ $\Lambda = e^x(e^x - 1)\partial_y \wedge \partial_z$
$((IV, -\tilde{X}^1), (IV.ii, -bX_3))$	$E = be^x \partial_z$ $\Lambda = -e^x(e^x - 1)\partial_y \wedge \partial_z$
$((IV, -\tilde{X}^1), (VI_0.i, -X_3))$	$E = e^x \partial_z$ $\Lambda = -2ye^x \partial_y \wedge \partial_z$
$((IV, -\tilde{X}^1), (VI_a.i, -X_3))$	$E = e^x \partial_z$ $\Lambda = \frac{2}{a-1} ye^x \partial_y \wedge \partial_z$
$((IV, -\tilde{X}^1), (VI_a.ii, -X_3))$	$E = e^x \partial_z$ $\Lambda = -\frac{2}{a+1} ye^x \partial_y \wedge \partial_z$
$((V, -2\tilde{X}^1), (V.i, -2X_2 - 2X_3))$	$E = 2e^x \partial_y + 2e^x \partial_z$ $\Lambda = e^x(1 - e^x)\partial_x \wedge \partial_y$ $+e^x(1 - e^x)\partial_x \wedge \partial_z$ $+e^{2x}(z - y)\partial_y \wedge \partial_z$
$((V, -\frac{2a}{a-1}\tilde{X}^1), (VI_a.i, -\frac{2a}{a-1}X_3))$	$E = \frac{2a}{a-1} e^x \partial_z$ $\Lambda = (e^x - e^{\frac{2a}{a-1}x})\partial_x \wedge \partial_z$ $-ye^{\frac{2a}{a-1}x}\partial_y \wedge \partial_z$
$((V, -\frac{2a}{a+1}\tilde{X}^1), (VI_a.ii, -\frac{2a}{a+1}X_3))$	$E = \frac{2a}{a+1} e^x \partial_z$ $\Lambda = (e^x - e^{\frac{2a}{a+1}x})\partial_x \wedge \partial_z$ $-ye^{\frac{2a}{a+1}x}\partial_y \wedge \partial_z$
$((IV, -\tilde{X}^1), (IV.i, -bX_3))$	$E = be^x \partial_z$ $\Lambda = e^x(e^x - 1)\partial_y \wedge \partial_z$
$((IV, -\tilde{X}^1), (IV.ii, -bX_3))$	$E = be^x \partial_z$ $\Lambda = -e^x(e^x - 1)\partial_y \wedge \partial_z$
$((IV, -\tilde{X}^1), (VI_0.i, -X_3))$	$E = e^x \partial_z$ $\Lambda = -2ye^x \partial_y \wedge \partial_z$
$((IV, -\tilde{X}^1), (VI_a.i, -X_3))$	$E = e^x \partial_z$ $\Lambda = \dfrac{2}{a-1} ye^x \partial_y \wedge \partial_z$
$((IV, -\tilde{X}^1), (VI_a.ii, -X_3))$	$E = e^x \partial_z$ $\Lambda = -\dfrac{2}{a+1} ye^x \partial_y \wedge \partial_z$
$((V, -2\tilde{X}^1), (V.i, -2X_2 - 2X_3))$	$E = 2e^x \partial_y + 2e^x \partial_z$ $\Lambda = e^x(1 - e^x)\partial_x \wedge \partial_y$ $+e^x(1 - e^x)\partial_x \wedge \partial_z$ $+e^{2x}(z - y)\partial_y \wedge \partial_z$

$((V, -\frac{2a}{a-1}\tilde{X}^1), (VI_a.i, -\frac{2a}{a-1}X_3))$	$E = \frac{2a}{a-1}e^x\partial_z$ $\Lambda = (e^x - e^{\frac{2a}{a-1}x})\partial_x \wedge \partial_z$ $\quad -ye^{\frac{2a}{a-1}x}\partial_y \wedge \partial_z$
$((V, -\frac{2a}{a+1}\tilde{X}^1), (VI_a.ii, -\frac{2a}{a+1}X_3))$	$E = \frac{2a}{a+1}e^x\partial_z$ $\Lambda = (e^x - e^{\frac{2a}{a+1}x})\partial_x \wedge \partial_z$ $\quad -ye^{\frac{2a}{a+1}x}\partial_y \wedge \partial_z$
$((VI_0, \tilde{X}^3), (III.vii, -X_1 - X_2))$	$E = \partial_x + \partial_y$ $\Lambda = (y - x)\partial_x \wedge \partial_y$ $\quad +(1 - e^{-2z})\partial_x \wedge \partial_z$ $\quad +(1 - e^{-2z})\partial_y \wedge \partial_z$
$((VI_0, \tilde{X}^3), (III.ix, -X_1))$	$E = \partial_x$ $\Lambda = (1 + y - e^{-z})\partial_x \wedge \partial_y$ $\quad +e^{-z}\sinh(z)\partial_x \wedge \partial_z$ $\quad +(1 - e^{-z}\cosh(z))\partial_y \wedge \partial_z$
$((VI_0, \tilde{X}^3), (VI_0.ii, -X_1 + X_2))$	$E = \partial_x - \partial_y$ $\Lambda = 2(x + y)\partial_x \wedge \partial_y$
$((VI_0, -2\tilde{X}^3),$ $(VI_0.ii, 2X_1 - 2X_2))$	$E = -2\partial_x + 2\partial_y$ $\Lambda = -(x + y)\partial_x \wedge \partial_y$ $\quad +(1 - e^{3z})\partial_x \wedge \partial_z$ $\quad -(1 - e^{3z})\partial_y \wedge \partial_z$
$((VI_0, \tilde{X}^3), (VI_a.iii, -X_1 + X_2))$	$E = \partial_x - \partial_y$ $\Lambda = -\frac{2}{a-1}(x + y)\partial_x \wedge \partial_y$
$((VI_0, \tilde{X}^3), (VI_a.iv, -X_1 + X_2))$	$E = \partial_x - \partial_y$ $\Lambda = \frac{2}{a+1}(x + y)\partial_x \wedge \partial_y$
$((VI_0, \frac{2}{a-1}\tilde{X}^3),$ $(VI_a.iii, -\frac{2}{a-1}(X_1 - X_2)))$	$E = \frac{2}{a-1}\partial_x - \frac{2}{a-1}\partial_y$ $\Lambda = -(x + y)\partial_x \wedge \partial_y$ $\quad +(1 - e^{\frac{a-3}{a-1}z})\partial_x \wedge \partial_z$ $\quad -(1 - e^{\frac{a-3}{a-1}z})\partial_y \wedge \partial_z$
$((VI_0, -\frac{2}{a+1}\tilde{X}^3),$ $(VI_a.iv, \frac{2}{a+1}(X_1 - X_2)))$	$E = -\frac{2}{a+1}\partial_x + \frac{2}{a+1}\partial_y$ $\Lambda = -(x + y)\partial_x \wedge \partial_y$ $\quad +(1 - e^{\frac{a+3}{a+1}z})\partial_x \wedge \partial_z$ $\quad -(1 - e^{\frac{a+3}{a+1}z})\partial_y \wedge \partial_z$
$((VI_a, -(a-1)\tilde{X}^1),$ $(III.ii, -\frac{a-1}{a+1}(X_2 + X_3)))$	$E = \frac{a-1}{a+1}e^{(a+1)x}\partial_y + \frac{a-1}{a+1}e^{(a+1)x}\partial_z$ $\Lambda = \frac{1}{a+1}(e^{(a+1)x} - e^{(a-1)x})\partial_x \wedge \partial_y$ $\quad +\frac{1}{a+1}(e^{(a+1)x} - e^{(a-1)x})\partial_x \wedge \partial_z$ $\quad +\frac{1}{a+1}e^{(a-1)x}(1 - a)(y - z)\partial_y \wedge \partial_z$

$((VI_a, -(a+1)\tilde{X}^1), (III.v, \frac{1}{a-1}(X_2 - aX_3)))$	$E = \frac{1}{a-1}e^{ax}(-\cosh(x) + a\sinh(x))\partial_y + \frac{1}{a-1}e^{ax}(-\sinh(x) + a\cosh(x))\partial_z$ $\Lambda = \frac{1}{a-1}e^{ax}\sinh(x)\partial_x \wedge \partial_y + \frac{1}{a-1}(e^{ax}\cosh(x) - e^{(a+1)x})\partial_x \wedge \partial_z + \frac{1}{a-1}(e^{2ax} - e^{(a+1)x}(1 + ay + z))\partial_y \wedge \partial_z$
$((VI_a, -(a-1)\tilde{X}^1), (III.v, \frac{1}{a+1}(X_2 - aX_3)))$	$E = \frac{1}{a+1}e^{ax}(-\cosh(x) + a\sinh(x))\partial_y + \frac{1}{a+1}e^{ax}(-\sinh(x) + a\cosh(x))\partial_z$ $\Lambda = \frac{1}{a+1}e^{ax}\sinh(x)\partial_x \wedge \partial_y + \frac{1}{a+1}(e^{ax}\cosh(x) - e^{(a-1)x})\partial_x \wedge \partial_z + \frac{1}{a+1}(e^{2ax} - e^{(a-1)x}(1 + ay + z))\partial_y \wedge \partial_z$
$((VI_a, -(a+1)\tilde{X}^1), (III.x, -\frac{a+1}{a-1}(X_2 - X_3)))$	$E = \frac{a+1}{a-1}e^{(a-1)x}\partial_y - \frac{a+1}{a-1}e^{(a-1)x}\partial_z$ $\Lambda = \frac{1}{a-1}(e^{(a-1)x} - e^{(a+1)x})\partial_x \wedge \partial_y + \frac{1}{a-1}(e^{(a+1)x} - e^{(a-1)x})\partial_x \wedge \partial_z + \frac{1}{a-1}e^{(a+1)x}(1+a)(y+z)\partial_y \wedge \partial_z$
$((VI_a, -(a+1)\tilde{X}^1), (VI_b.v, -X_2 - X_3))$	$E = e^{(a+1)x}\partial_y + e^{(a+1)x}\partial_z$ $\Lambda = \frac{2}{(b-1)}e^{(a+1)x}(y - z)\partial_y \wedge \partial_z$
$((VI_a, -(a+1)\tilde{X}^1), (VI_b.vi, -X_2 + X_3))$	$E = e^{(a+1)x}\partial_y + e^{(a+1)x}\partial_z$ $\Lambda = -\frac{2}{(b+1)}e^{(a+1)x}(y - z)\partial_y \wedge \partial_z$
$((VI_a, -(a-1)\tilde{X}^1), (VI_b.vii, -X_2 + X_3))$	$E = e^{(a-1)x}\partial_y - e^{(a-1)x}\partial_z$ $\Lambda = -\frac{2}{(b-1)}e^{(a-1)x}(y + z)\partial_y \wedge \partial_z$
$((VI_a, -(a-1)\tilde{X}^1), (VI_b.viii, -X_2 + X_3))$	$E = e^{(a-1)x}\partial_y - e^{(a-1)x}\partial_z$ $\Lambda = \frac{2}{(b+1)}e^{(a-1)x}(y + z)\partial_y \wedge \partial_z$
$((VI_a, -\frac{2(ab+1)}{b-1}\tilde{X}^1), (VI_b.v, -\frac{2(ab+1)}{(a+1)(b-1)}(X_2 + X_3)))$	$E = \frac{2(ab+1)}{(a+1)(b-1)}e^{(a+1)x}\partial_y + \frac{2(ab+1)}{(a+1)(b-1)}e^{(a+1)x}\partial_z$ $\Lambda = \frac{1}{a+1}(e^{(a+1)x} - e^{\frac{2(ab+1)}{b-1}x})\partial_x \wedge \partial_y + \frac{1}{a+1}(e^{(a+1)x} - e^{\frac{2(ab+1)}{b-1}x})\partial_x \wedge \partial_z + \frac{1}{a+1}e^{\frac{2(ab+1)}{b-1}x}(1-a)(y-z)\partial_y \wedge \partial_z$
$((VI_a, -\frac{2(ab-1)}{b+1}\tilde{X}^1), (VI_b.vi, -\frac{2(ab-1)}{(a+1)(b+1)}(X_2 + X_3)))$	$E = \frac{2(ab-1)}{(a+1)(b+1)}e^{(a+1)x}\partial_y + \frac{2(ab-1)}{(a+1)(b+1)}e^{(a+1)x}\partial_z$ $\Lambda = \frac{1}{a+1}(e^{(a+1)x} - e^{\frac{2(ab-1)}{b+1}x})\partial_x \wedge \partial_y + \frac{1}{a+1}(e^{(a+1)x} - e^{\frac{2(ab-1)}{b+1}x})\partial_x \wedge \partial_z + \frac{1}{a+1}e^{\frac{2(ab-1)}{b+1}x}(1-a)(y-z)\partial_y \wedge \partial_z$

$((VI_a, -\frac{2(ab-1)}{b-1}\tilde{X}^1),$ $(VI_b.vii, -\frac{2(ab-1)}{(a1)(b-1)}(X_2 - X_3)))$	$E = \frac{2(ab-1)}{(a-1)(b-1)}e^{(a-1)x}\partial_y$ $-\frac{2(ab-1)}{(a-1)(b-1)}e^{(a-1)x}\partial_z$ $\Lambda = \frac{1}{a-1}(e^{(a-1)x} - e^{\frac{2(ab-1)}{b-1}x})\partial_x \wedge \partial_y$ $+\frac{1}{a-1}(-e^{(a-1)x} + e^{\frac{2(ab-1)}{b-1}x})\partial_x \wedge \partial_z$ $+\frac{1}{a-1}e^{\frac{2(ab-1)}{b-1}x}(1+a)(y+z)\partial_y \wedge \partial_z$
$((VI_a, -\frac{2(ab+1)}{b+1}\tilde{X}^1),$ $(VI_b.viii,$ $-\frac{2(ab+1)}{(a-1)(b+1)}(X_2 - X_3)))$	$E = \frac{2(ab+1)}{(a-1)(b+1)}e^{(a-1)x}\partial_y$ $-\frac{2(ab+1)}{(a-1)(b+1)}e^{(a-1)x}\partial_z$ $\Lambda = \frac{1}{a-1}(e^{(a-1)x} - e^{\frac{2(ab+1)}{b+1}x})\partial_x \wedge \partial_y$ $+\frac{1}{a-1}(-e^{(a-1)x} + e^{\frac{2(ab+1)}{b+1}x})\partial_x \wedge \partial_z$ $+\frac{1}{a-1}e^{\frac{2(ab+1)}{b+1}x}(1+a)(y+z)\partial_y \wedge \partial_z$

Table 5.3: Reeb vector field and Jacobi bivector field related to real three-dimensional coboundary Jacobi–Lie bialgebras.

$((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$	Reeb vector field E and Jacobi bivector field Λ
$((I, 0), (V, -X_1))$	$E = \partial_x$ $\Lambda = 0$
$((II, -\tilde{X}^2 + \tilde{X}^3), (III, -2X_1))$	$E = 2\partial_x$ $\Lambda = (-1 + e^{y-z})\partial_x \wedge \partial_y$ $+(1 - e^{y-z})\partial_x \wedge \partial_z$
$((III, -\tilde{X}^2 + \tilde{X}^3),$ $(III.iii, X_2 + X_3))$	$E = -e^{2x}\partial_y - e^{2x}\partial_z$ $\Lambda = (-e^{2x} + e^{y-z})\partial_y \wedge \partial_z$
$((III, 0), (III.x, -X_2 + X_3))$	$E = \partial_y - \partial_z$ $\Lambda = (y+z)\partial_y \wedge \partial_z$
$((IV, -\tilde{X}^1), (III.v, -X_3))$	$E = e^x\partial_z$ $\Lambda = -(y + 1 - e^x)e^x\partial_y \wedge \partial_z$
$((IV, -2\tilde{X}^1), (V.ii, -2X_3))$	$E = 2e^x\partial_z$ $\Lambda = (e^x - e^{2x})\partial_x \wedge \partial_z - ye^{2x}\partial_y \wedge \partial_z$
$((IV, -\frac{2a}{a-1}\tilde{X}^1), (VI_a.i, -\frac{2a}{a-1}X_3))$	$E = \frac{2a}{a-1}e^x\partial_z$ $\Lambda = (e^x - e^{\frac{2a}{a-1}x})\partial_x \wedge \partial_z - ye^{\frac{2a}{a-1}x}\partial_y \wedge \partial_z$
$((IV, -\frac{2a}{a+1}\tilde{X}^1),$ $(VI_a.ii, -\frac{2a}{a+1}X_3))$	$E = \frac{2a}{a+1}e^x\partial_z$ $\Lambda = (e^x - e^{\frac{2a}{a+1}x})\partial_x \wedge \partial_z - ye^{\frac{2a}{a+1}x}\partial_y \wedge \partial_z$
$((V, -\tilde{X}^1), (VI_0.i, -X_3))$	$E = e^x\partial_z$ $\Lambda = -2ye^x\partial_y \wedge \partial_z$

$((V, -\tilde{X}^1), (VI_a.i, -X_3))$	$E = e^x \partial_z$ $\Lambda = \frac{2}{a-1} y e^x \partial_y \wedge \partial_z$
$((V, -\tilde{X}^1), (VI_a.ii, -X_3))$	$E = e^x \partial_z$ $\Lambda = -\frac{2}{a+1} y e^x \partial_y \wedge \partial_z$
$((VI_0, \tilde{X}^3), (III.viii, -X_1 + X_2))$	$E = \partial_x - \partial_y$ $\Lambda = (x + y)\partial_x \wedge \partial_y$
$((VI_a, -(a+1)\tilde{X}^1),$ $(III.ii, -X_2 - X_3))$	$E = e^{(a+1)x}\partial_y + e^{(a+1)x}\partial_z$ $\Lambda = -(y - z)e^{(a+1)x}\partial_y \wedge \partial_z$
$((VI_a, -(a-1)\tilde{X}^1),$ $(III.x, -X_2 + X_3))$	$E = e^{(a-1)x}\partial_y + e^{(a-1)x}\partial_z$ $\Lambda = (y + z)e^{(a-1)x}\partial_y \wedge \partial_z$

Acknowledgments. The authors are greatly indebted to Professor J.F. Cariñena for his patience in careful reading the manuscript and helpful suggestions that improved significantly the presentation of the paper.

References

[1] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, Addison–Wesley, Redwood City, 1987.

[2] H. Amirzadeh-Fard, Gh. Haghighatdoost, P. Kheradmandynia, and A. Rezaei-Aghdam, *Jacobi structures on real two- and three-dimensional Lie groups and their Jacobi–Lie systems*, Theoret. and Math. Phys. **205** (2020), 1393–1410.

[3] R. Angelo, E. Duzzioni, and A. Ribeiro, *Integrability in time-dependent systems with one degree of freedom*, J. Phys. A **45** (2012), 055101.

[4] A. Ballesteros, J.F. Cariñena, F.J. Herranz, J. de Lucas, and C. Sardón, *From constants of motion to superposition rules for Lie–Hamilton systems*, J. Phys. A **46** (2013), 285203.

[5] J.F. Cariñena, J. Grabowski, and G. Marmo, *Lie–Scheffers Systems: a Geometric Approach*, Bibliopolis, Naples, 2000.

[6] J.F. Cariñena, J. Grabowski, and G. Marmo, *Superposition rules, Lie theorem and partial differential equations*, Rep. Math. Phys. 60 (2007), 237–258.

[7] J.F. Cariñena, J. de Lucas, and C. Sardón, *Lie–Hamilton systems: theory and applications*, Int. J. Geom. Methods Mod. Phys. **10** (2013), 1350047.

[8] Z. Fiala, *Evolution equation of Lie-type for finite deformations, time-discrete integration, and incremental methods*, Acta Mech. **226** (2015), 17–35.

[9] A. Guldberg, *Sur les équations différentielles ordinaires qui possèdent un système fondamental d'intégrales*, C.R. Math. Acad. Sci. Paris **116** (1893), 964–965.

[10] F.J. Herranz, J. de Lucas, and C. Sardón, *Jacobi–Lie systems: Fundamentals and low-dimensional classification*, in: Dynamical Systems and Differential Equations, AIMS Proceedings 2015, Proceedings of the 10th AIMS International Conference (Madrid, Spain), Discrete Contin. Dyn. Syst. (Suppl.), 605–614 (2015).

[11] D. Iglesias and J. C. Marrero, *Generalized Lie bialgebras and Jacobi structures on Lie groups*, Israel J. Math. **133** (2003) 285–320.

[12] A. Kirillov, *Local Lie algebras*, Russ. Math. Surv. **31** (1976), 55–76.

[13] L. Köningsberger, *Über die einer beliebigen differentialgleichung erster ordnung angehörigen selbständigen transcendenten*, Acta Math. **3** (1883), 1–48.

[14] Y. Kosmann-Schwarzbach, *Lie Bialgebras, Poisson Lie Groups and Dressing Transformations, Integrability of Nonlinear Systems*, Lecture Notes in Physics, **638**, Springer-Verlag, 2004, 107–173.

[15] A. Lichnerowicz, *Les variétés de Jacobi et leurs algèbres de Lie associées*, J. Math. Pures et appl. **57** (1978), 453–488.

[16] S. Lie and G. Scheffers, *Vorlesungen über Continuierliche Gruppen Mit Geometrischen und Anderen Anwendungen*, Teubner, Leipzig, 1893.

[17] J. de Lucas and C. Sardón, *A Guide to Lie Systems with Compatible Geometric Structures*, World Scientific, Singapore, 2020.

[18] R.O. Popovych, V.M. Boyko, M.O. Nesterenko, and M.W. Lutfullin, *Realization of real low dimensional Lie algebra,* J. Phys. A: Math. Gen. **36** (2003), 7337–7360.

[19] A. Rezaei-Aghdam, M. Hemmati, and A. R. Rastkar, *Classification of real three-dimensional Lie bialgebras and their Poisson–Lie groups*, J. Phys. A: Math. Gen. **38** (2005) 3981–3994.

[20] A. Rezaei-Aghdam and M. Sefid, *Classical r-matrices of real low-dimensional Jacobi-Lie bialgebras and their Jacobi–Lie groups*, Int. J. Geom. Methods Mod. Phys. **13** (2016) 1650087.

[21] A. Rezaei-Aghdam and M. Sephid, *Classification of real low dimensional Jacobi(generalized)–Lie bialgebras*, Int. J. Geom. Methods Mod. Phys. **14** (2017), 1750007.

[22] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Progress in Mathematics, **118**, Birkhäuser Verlag, Basel, 1994.

[23] E. Vessiot, *Sur une classe d'équations différentielles*, Ann. Sci. École Norm. Sup. (3) **10** (1893), 53–64.

[24] P. Winternitz, *Lie groups and solutions of nonlinear differential equations*, in: Nonlinear Phenomena, Lecture Notes in Phys, **189**, Springer, Berlin, 1983, 263–331.

Received October 15, 2020, revised January 19, 2021.

H. Amirzadeh-Fard,

Department of Mathematics, Azarbaijan Shahid Madani University, 53714-161, Tabriz, Iran,

E-mail: h.amirzadehfard@azaruniv.ac.ir

Gh. Haghighatdoost,

Department of Mathematics, Azarbaijan Shahid Madani University, 53714-161, Tabriz, Iran,

E-mail: e-mail: gorbanali@azaruniv.ac.ir

A. Rezaei-Aghdam,

Department of Physics, Azarbaijan Shahid Madani University, 53714-161, Tabriz, Iran,

E-mail: e-mail: rezaei-a@azaruniv.ac.ir

Гамільтонові системи Якобі–Лі на дійсних маловимірних групах Якобі–Лі та їх симетрії Лі

H. Amirzadeh-Fard, Gh. Haghighatdoost, and A. Rezaei-Aghdam

Ми вивчаємо гамільтонові системи Якобі–Лі, які допускають алгебри Лі Вессіо–Гульдберга гамільтонових векторних полів пов'язаних зі структурами Якобі на дійсних маловимірних групах Якобі–Лі. Також ми знаходимо всі можливі приклади гамільтонових систем Якобі–Лі на дійсних дво- і тривимірних групах Якобі–Лі. Наостанок ми представляємо симетрії Лі гамільтонових систем Якобі–Лі на дійсній тривимірній групі Лі $SL(2, \mathbb{R})$.

Ключові слова: група Якобі–Лі, многовид Якобі, система Лі, гамільтонова система Якобі–Лі, симетрія Лі